

# Functional Registration and Local Variations

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**Abstract:** We study the problem of nonparametric registration of functional data that have been subjected to random deformation (warping) of their time scale. The separation of this phase variation (“horizontal” variation) from the amplitude variation (“vertical” variation) is crucial in order to properly conduct further analyses, which otherwise can be severely distorted. We determine precise conditions under which the two forms of variation are identifiable, under minimal assumptions on the form of the warp maps. We show that these conditions are sharp, by means of counterexamples. We then propose a nonparametric registration method based on a “local variation measure”, which bridges functional registration and optimal transportation. Our method is proven to consistently estimate the warp maps from discretely observed data, without requiring any penalisation or tuning on the warp maps themselves. This circumvents the problem of over/under-registration often encountered in practice. Similar results hold in the presence of measurement error, with the addition of a pre-processing smoothing step. We carry out a detailed theoretical investigation of the strong consistency and the weak convergence properties of the resulting functional estimators, establishing rates of convergence. We also give a theoretical study of the impact of deviating from the identifiability conditions, quantifying it in terms of the spectral gap of the amplitude variation. Numerical experiments demonstrate the good finite sample performance of our method, and the methodology is further illustrated by means of a data analysis.

**Keywords:** Identifiability, Optimal Transportation, Registration, Total variation, Warping

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## 1. Introduction

Functional observations can fluctuate around their mean structure in broadly two ways: (a) amplitude variation, and (b) phase variation. The first type of variation is analysed using functional principal component analysis, which stratifies the variation in amplitude (or variation in the “vertical axis”) across the different eigenfunctions of the covariance operator of the underlying distribution. The second kind of variation, if present, is more subtle and can drastically distort the analysis of a functional dataset. It typically manifests itself in functional data representing physiological processes or physical motion, and consists in deformations of the time scale of the functional data (or variation in the “horizontal axis”), associating to each observation its own unobservable time scale resulting from a transformation of the original time scale by a time warp. Specifically, instead of observing curves  $\{X_i(t) : [0, 1] \rightarrow \mathbb{R}\}_{i=1}^n$ , one actually observes warped versions  $\tilde{X}_i = X_i \circ T_i^{-1}$ , where the  $T_i$ ’s are unobservable (random) homeomorphisms termed *warp maps*. In the presence of phase variation, the mean of the warped data conditional on the warping,  $E(X_i|T_i) = \mu \circ T_i^{-1}$ , is a distortion of the true mean  $\mu$  by the warp map. Failing to account for the time transformation will yield deformed mean estimates, converging to  $E[\mu \circ T_i^{-1}]$  rather than  $\mu$ . More dramatic still will be the effect on the estimation of the covariance of the latent process, inflating its essential rank, and yielding uninterpretable principal components. We refer to Section 2 in [Panaretos and Zemel \(2016\)](#) for a detailed discussion of these effects. Consequently, in the presence of phase variation in the data, the natural first step in the analysis should be to *register* the data, i.e., to simultaneously transform/synchronise the curves back to the *objective* time scale.

Owing to the rather complex nature of the registration problem, a variety of different assumptions on the nature of the warp maps  $T_i$  have been considered, and correspondingly a multitude of methods have been investigated: landmark based registration ([Kneip and Gasser, 1992](#)); template/target based registration ([Ramsay and Li, 1998](#)); registration using dynamic time warping ([Wang and Gasser, 1997, 1999](#)); registration based on local regression ([Kneip et al., 2000](#)); a “self-modelling” approach by [Gervini and Gasser \(2004\)](#) for warp maps expressible as linear combinations of B-splines; related registration procedures under assumptions on functional forms of the warp maps that result in a finite dimensional family of deformations ([Rønn, 2001](#); [Gervini and Gasser, 2005](#)); a functional convex synchronization approach to registration ([Liu and Müller, 2004](#)); registration using “moments” of the data curves ([James, 2007](#)); registration based on a parsimonious representation of the registered observations by the principal components ([Kneip and Ramsay, 2008](#)); pairwise registration of the warped functional data under monotone piecewise-linear warp maps ([Tang and Müller, 2008](#)); a joint amplitude-phase analysis with this pairwise registration procedure but considering step-function (thus finite dimensional) approximations of the warp maps using finite difference of their log-derivatives (centered log-ratio transform) ([Hadjipantelis et al., 2015](#)); registration when the warp maps are generated as compositions of elementary “warplets” ([Claeskens, Silverman and Slaets, 2010](#)); and registration using a warp-invariant metric between curves when the warp functions are diffeomorphisms on an interval ([Srivastava et al., 2011](#)). The above list is not exhaustive and we refer to [Marron et al. \(2015\)](#) for an overview and comparison of some of the registration procedures mentioned above.

Several of the above contributions consider the case when the warp maps are themselves random, and in such cases, a canonical set of assumptions is usually required:

- (a)  $T$  is an increasing homeomorphism with probability one, and
- (b)  $E(T) = Id$ , where  $Id$  is the identity map,  $Id(x) = x$ .

The first assumptions rules out “time-reversal” or “time-jumps”, while the second disallows an overall speed-up or slow-down of time. Further to these natural assumptions, additional smoothness and structural assumptions are typically also imposed on  $T$  (e.g. Rønn (2001), Gervini and Gasser (2004, 2005), Tang and Müller (2008) and Claeskens, Silverman and Slaets (2010)) which require tuning parameters to be selected. However, it is unclear whether these additional assumptions are truly necessary for identifiability to hold. It is natural to ask what assumptions must one minimally impose on the latent functional data generating process so that the registration problem be identifiable only under conditions (a) and (b) on the warp maps.

There seems to be a consensus that identifiability (and hence consistency) rests crucially on an implicit assumption that either phase variation or amplitude variation *is of low rank*. In other words, that one or the other type of variation is *dominant*. This is reflected directly or indirectly in many of the earlier contributions in the literature, whether by explicit model assumptions, or by the asymptotic regime considered. Often, either the warp maps are assumed to be essentially parametric or the data generating process is low rank, or both e.g., Rønn (2001), Gervini and Gasser (2005), James (2007) and Claeskens, Silverman and Slaets (2010). Variants of the model

$$X_i(t) = \xi_i \phi(t) + \delta \epsilon_i(t), \quad i = 1, 2, \dots, n \quad (1)$$

for the latent process are often encountered in the literature, with  $\phi$  a unit norm deterministic function,  $\xi$  a random variable,  $\epsilon_i(t)$  a zero-mean error term of unit variance (i.e.  $E\|\epsilon_i\|_2^2 = 1$ ), and  $\delta^2$  small relative to  $\text{var}\{\xi_i\}$ . In other words, it is postulated that if it were not for phase variation, important landmark features such as peaks and valleys of the latent process would not drastically change from realisation to realisation. Gervini and Gasser (2004) assumed this representation, but imposing rather restrictive conditions on the  $\xi_i$ ’s and the  $T_i$ ’s. Tang and Müller (2008) assumed that the  $T_i$  admit a  $p$ -dimensional representation in a known basis,  $\xi_i = 1$  for all  $i$ , and additionally that  $\delta \rightarrow 0$  as  $n \rightarrow \infty$  to ensure asymptotically identifiable (consistent) registration. Similar models were also considered by Wang and Gasser (1997, 1999) and Srivastava et al. (2011). Note that Model (1) is a rank one model for the latent process when one has exactly  $\delta = 0$  rather than just  $\delta^2 \ll \text{var}\{\xi_i\}$ . This model also fits with the landmark principle of registration (Kneip and Gasser, 1992), which essentially stipulates that the true curves have similar shape (thus having the same landmarks) but possibly differ in their amplitude component.

### 1.1. Our Contributions

Our contributions in this paper are twofold. Firstly, we elucidate the link between the identifiability issue for a general registration problem, the rank of the latent model, and the warp map structure as reflected in assumptions (a) and (b) (Section 2). We prove that the registration problem is identifiable when the amplitude variation is exactly of rank 1 (Theorem 1). Conversely, unless more restrictive structural conditions are imposed on the warp maps, we show by means of counterexamples that the rank 1 assumption is also *necessary* – even a low rank assumption on the warp maps will not do.

Secondly, we develop methodology to address the problem of *nonparametric* and *consistent* recovery of the warp maps from discretely warped curves, *without structural assumptions on the warp maps* further to (a) and (b), and *without any penalisation or tuning parameters* related to the warp maps themselves. Minimal structural assumptions are particularly desirable since, in practice, one rarely has more detailed insights regarding the underlying warping phenomenon. And circumventing penalisation/tuning has the

important practical advantage that there is no danger of “over-registering” (overfitting) the data, on account of the tuning of a penalty on the registration maps.

The key to our identifiability and methodological results is the novel use of a criterion that measures the local amount of deformation of the time scale (Section 3). Specifically, we introduce the *local variation measure* of  $X$ , with associated cumulative distribution  $J_X(t) = \int_0^t |X'(u)| du$ , which reflects how the total variation of the curve is distributed on the real axis. We show that the effect of the time deformation on the local variation distribution has a transparent interpretation in terms of optimal transportation (see, e.g., Villani (2003)). Our approach exploits this connection in order to deduce identifiability and to estimate the unobservable warp maps and register the functional data. Indeed, it is precisely the structure of optimal transportation that exempts us from the need of additional smoothness/structural conditions on the warp maps  $T$ , and consequently from the need to introduce registration tuning parameters – even when the curves are observed over a discrete grid<sup>1</sup>. Although our procedure involves derivatives, we actually *do not need to estimate any derivatives* from discretely observed data if there is no measurement error, as we can exploit an equivalent definition of total variation using finite differences over partitions of the domain. If there is measurement error, then a pre-processing smoothing step is required, but still no additional penalisation of the registration maps is necessary.

We carry out a complete asymptotic analysis in all possible observation regimes: complete observation, discrete observation without error, and measurement error contamination. In all cases, we prove that the nonparametric estimators obtained are consistent as the number of observations grows, and the measurement grid becomes dense, and additionally derive rates of convergence and weak convergence for all the quantities involved (Section 4, Theorems 2, 3, 4, 5). We also investigate the potential impact of model misspecification (departures from the identifiable regime) in Section 4.2, where we derive theoretical results quantifying the asymptotic amount of bias incurred in terms of the spectral gap of the amplitude variation (Theorem 6).

The finite sample performance of our methodology is studied in Section 5, for all possible observation regimes. We further numerically probe the impact misspecification, and quantify the stability of our method depending on the degree of misspecification. The method is further illustrated by analysis of a functional dataset of *Triboleum* beetle larvae growth curves, in Section 6. The proofs of all the theorems are collected in the Appendix.

## 2. Model Identifiability

Recall that a “standard model” for the latent process prior to warping (Equation 1) takes the general form

$$X(t) = \xi\phi(t) + \delta\epsilon(t).$$

This, depending on the constraints imposed on the random variable  $\xi$  and the scalar  $\delta$ , can be of arbitrarily large rank. Usually  $\text{var}\{\xi\}$  is expected to be the dominant effect relative to  $\delta$  (i.e.  $\delta^2 \ll \text{var}\{\xi\}$ ), corresponding to an “approximately rank 1” model. We now give sufficient conditions on the “standard

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<sup>1</sup>Of course, once the warp maps are estimated, one would have to smooth the warped discrete data in order to register them, since the warped data are not observed at all points of their domain. And, if there is measurement error in the observations, then some pre-smoothing will be needed. But in either case, this smoothing will be on the data itself (either as a pre-processing or post-processing step), and no smoothing penalties or structural assumptions will be required on the registration maps themselves.

model" so that identifiability will hold, without any assumptions on  $T$  further to the minimal assumptions (a) and (b). In simple terms, the process must be *exactly* of rank 1: either  $\delta$  must be zero, or  $\epsilon(t)$  a deterministic function in the span of  $\phi$ .

**Theorem 1** (Identifiability). *Let  $\{X_1, X_2\}$  be a random elements in  $C^1[0, 1]$  of rank one, i.e.,  $X_i(t) = \xi_i \phi_i(t)$  for deterministic functions  $\phi_i$  with  $\|\phi_i\|_2 = 1$ , and with  $\phi'_i$  vanishing on at most a countable set. Assume that  $\{T_1, T_2\}$  are strictly increasing homeomorphisms in  $C^1[0, 1]$ , and such that  $E(T_i) = Id$ . Write  $\tilde{X}_i = X_i(T_i^{-1}(t))$ . Then,*

$$\tilde{X}_1 \stackrel{d}{=} \tilde{X}_2 \iff \left\{ T_1 \stackrel{d}{=} T_2, \quad \phi_1 = \pm \phi_2, \quad \xi_1 = \pm \xi_2 \right\}.$$

The next counterexample shows that the rank 1 condition is necessary, unless more than (a) and (b) is assumed on  $T$ .

**Counterexample 1** (Necessity of rank 1 condition). We will show that the same process can arise in two ways: (i) as a rank one process subjected to warping by a non-trivial warp map  $T$  satisfying (a) and (b), and (ii) a rank two process with no warping (i.e., with identity warp map, which obviously satisfies (a) and (b)).

Define

$$f(t) = t^2 \quad \& \quad g(t) = (4t - t^2)/3, \quad t \in [0, 1].$$

Take  $\xi$  to be a standard Gaussian random variable and  $\phi(t) = t/\sqrt{3}$  for  $t \in [0, 1]$ . Now define a random warp map  $T$  such that  $P[T = f] = 1 - P[T = g] = 1/4$ . Then  $T$  satisfies (a), and  $E(T) = f/4 + 3g/4 = Id$ , so (b) is also satisfied. Now,

$$\tilde{X} = \xi \phi \circ T^{-1} = \xi_1 T^{-1} = \xi_1 (f^{-1}U + g^{-1}(1 - U)),$$

where  $U$  is a Bernoulli random variable with success probability  $1/4$  and  $\xi_1 = \xi/\sqrt{3}$ . Let  $V = \xi_1 U$  and  $W = \xi_1(1 - U)$  so that

$$\tilde{X} = Vf_1 + Wg_1,$$

where  $f_1(t) = f^{-1}(t) = \sqrt{t}$  and  $g_1(t) = g^{-1}(t) = 2 - \sqrt{4 - 3t}$ ,  $t \in [0, 1]$ . It is easy to check that  $Var(V) = 1/12$ ,  $Var(W) = 1/4$  and  $Cov(V, W) = 0$ . Further, it is easy to show that  $f_1$  and  $g_1$  are linearly independent. Consequently, we may define a new process  $Y = Vf_1 + Wg_1$ , which is a rank two process. Define  $\tilde{Y} = Y \circ Id^{-1} = Y$ . Then,  $\tilde{X} \stackrel{d}{=} \tilde{Y}$  (in fact  $\tilde{X} = \tilde{Y}$ ) but they have been generated using two different warp maps, namely  $T$  and  $Id$ , which of course do not have the same distribution

Note that the model for  $X$  considered in Theorem 1 implies that  $\mu(t) := E[X(t)] = \phi(t)E(\xi)$ , i.e.,  $\mu$  is constrained to lie in the span of  $\phi$ . In particular the model  $X = \mu + \xi\phi$  is *not* identifiable subject to warping unless  $\mu \in \text{span}\{\phi\}$ . A counterexample can be constructed along the same lines as above.

Finally, one might also wonder, whether at least the assumption that  $E[T] = Id$  can be relaxed in the context of the rank 1 model. The next counterexample shows that this is not the case either.

**Counterexample 2** (Necessity of  $E(T) = Id$ ). Suppose that  $E(T) = f_0$  with  $f_0 \neq Id$  and  $f_0$  being a strictly increasing homeomorphism on  $[0, 1]$ . Define  $S = T \circ f_0^{-1}$ . It follows that  $E(S) = Id$ . Now  $\tilde{X} = \xi \phi \circ T^{-1} = \xi \phi \circ f_0^{-1} \circ S^{-1} = \xi \phi_0 \circ S^{-1}$ , where  $\phi_0 = \phi \circ f_0^{-1}$ . Let  $c_0 = \|\phi_0\|_2$ . Define  $\xi_0 = c_0 \xi$  and  $\phi_1 = \phi_0/c_0$ . Then,  $\|\phi_1\| = 1$ . So the resulting processes have the same distribution (actually the processes are equal) but have been generated using different warp maps  $S$  and  $T$ , which do not have the same distribution because they have different means.

In light of Theorem 1, we will postulate the following model:

**Assumptions 1** (Identifiable Phase Variation Model). *The warped random function  $\tilde{X} : [0, 1] \rightarrow \mathbb{R}$  arises as*

$$\tilde{X}(t) = X(T^{-1}(t)) = \xi\varphi(T^{-1}(t)), \quad (2)$$

where:

(M1)  $\xi$  is a real-valued random variable of finite variance.

(M2)  $\varphi \in C^1([0, 1])$  is a deterministic function of unit  $L^2$ -norm, whose derivative vanishes at most on a countable subset of  $[0, 1]$ .

(M3)  $T$  is a strictly increasing random  $C^1$ -homeomorphism on  $[0, 1]$  such that  $\mathbb{E}[T] = Id$ .

With model identifiability being established, we now turn to nonparametric methods of estimation. For these, we will require the notion of local variation measure, introduced in the next section.

### 3. Methodology

Recall that the total variation of a continuous function  $h(x) : [0, 1] \rightarrow \mathbb{R}$  measures the total distance swept by the ordinate  $y = h(x)$  of its graph, as the abscissa  $x$  moves from 0 to 1. By distorting functions “in the  $x$ -domain” through an increasing homeomorphism, phase variation will not affect the total amount of variation accrued over the interval  $[0, 1]$ . However, it will *redistribute* this total variation over the subintervals of  $[0, 1]$ . This redistribution can be measured by focussing on *local variation*:

**Definition 1** (Local Variation Distribution). *Given any real function  $h \in C([0, 1])$ , we define*

$$J_h(t) = \sup_{K \in \mathcal{K}_t} \sum_{k=0}^{|K|} |h(\tau_{k+1}) - h(\tau_k)| \quad (3)$$

where  $K_t = \{\tau_0, \tau_1, \dots, \tau_{|K|}\}$  is a partition of  $[0, t]$  and  $\mathcal{K}_t$  is the collection of all finite partitions of  $[0, t]$ . When  $h \in C^1([0, 1])$ , it holds that

$$J_h(t) = \int_0^t |h'(u)| du. \quad (4)$$

Noting that  $J_h(1)$  is the total variation of  $h$ , we define the local variation distribution as

$$F_h(t) = J_h(t) / J_h(1).$$

The main motivation behind our approach is that warping affects the local variation of the underlying process in a rather predictable manner – one that can be used as a basis for estimation:

**Lemma 1** (Local Variations and Warp Maps). *In the context of Model 2, let  $F$  and  $\tilde{F}$  denote the local variation distributions associated with functions  $X$  and  $\tilde{X}$ . Then, under assumptions (M1)-(M3),  $F$  and  $\tilde{F}$  are strictly monotone almost surely, and satisfy,*

$$F = F_\phi, \quad T = \tilde{F}^{-1} \circ F_\phi, \quad \& \quad \mathbb{E}(\tilde{F}^{-1}) = F_\phi^{-1}.$$

**Remark 1.** *In the language of optimal transportation, Lemma 1 simply says that the warp map pushes forward the original local variation distribution  $F (= F_\phi)$  to the warped local variation distribution  $\tilde{F}$ , and indeed optimally in terms of quadratic transportation cost. Furthermore, the equality  $\mathbb{E}(\tilde{F}^{-1}) = F_\phi^{-1}$  at the level of quantile functions is equivalent to stating that  $F_\phi (= F)$  is the Fréchet mean of the random distribution  $\tilde{F}$ , with respect to the 2-Wasserstein distance (Villani (2003)).*



Now consider an i.i.d. sample  $\{\tilde{X}_i : i = 1, 2, \dots, n\}$  of randomly warped functional data issued from Model 2. We wish to construct nonparametric estimators of the  $\{\phi_i\}_{i=1}^n$  and predictors of the  $\{T_i\}_{i=1}^n$  on the basis of  $\{\tilde{X}_i\}_{i=1}^n$ . The key to this will be to employ the Equation  $\mathbb{E}(\tilde{F}^{-1}) = F_\phi^{-1}$  in Lemma 1. We consider three different observation regimes on  $\{\tilde{X}_i\}_{i=1}^n$  in the next three sections: complete observation, discrete noiseless observation, and discrete observation with measurement error.

### 3.1. Fully Observed Functions

If the functions  $\{\tilde{X}\}$  are fully observed, we may proceed as follows:

Step 1: Estimate  $F_\phi$  by its natural (as per Lemma 1) estimator

$$\hat{F}_\phi = \left( n^{-1} \sum_{i=1}^n \tilde{F}_i^{-1} \right)^{-1},$$

noting that the  $\{\tilde{F}_i\}$  are immediately available by complete observation of the  $\{\tilde{X}_i\}$ .

Step 2: Predict the warp map  $T_i$  by  $\hat{T}_i = \tilde{F}_i^{-1} \circ \hat{F}_\phi$ , and the registration map  $T_i^{-1}$  by  $\hat{T}_i^{-1}$ .

Step 3: Register the observed warped functional data, by means of  $\hat{X}_i = \tilde{X}_i \circ \hat{T}_i$ .

Step 4: Compute the empirical covariance operator, say,  $\hat{\mathcal{K}}_r$  of the registered data  $\{\hat{X}_i\}$  and estimate  $\phi$  by the leading eigenfunction  $\hat{\phi}$  of  $\hat{\mathcal{K}}_r$  (as a convention, assume that this estimator is aligned with the true  $\phi$ , i.e.,  $\langle \hat{\phi}, \phi \rangle \geq 0$ ).

Step 5: Estimate  $\xi_i$  by  $\hat{\xi}_i = \langle \hat{X}_i, \hat{\phi} \rangle_2$ .

### 3.2. Discretely Observed Functions

In the discretely observed setting, the  $\tilde{X}_i$ 's are not fully observed. Instead, we observe point evaluations

$$\tilde{X}_{i,d} = (\tilde{X}_i(t_1), \tilde{X}_i(t_2), \dots, \tilde{X}_i(t_r))', \quad i = 1, \dots, n.$$

Here,  $0 \leq t_1 < t_2 < \dots < t_r \leq 1$  is a grid over  $[0, 1]$ , assumed asymptotically homogeneous in that  $\max_{1 \leq j \leq r-1} (t_{j+1} - t_j) = O(r^{-1})$  as  $r \rightarrow \infty$ . Note that since the observations have been generated from Model 2, we have

$$X_i(t_j) = \xi_i \phi(T_i^{-1}(t_j)).$$

The latent discrete process is denoted by  $X_{i,d} = (X_i(t_1), X_i(t_2), \dots, X_i(t_r))'$  and satisfies  $X_i(t_j) = \xi_i \phi(t_j)$ .

Our strategy will be to mimic Steps 1–5 from the fully observed setup. Since the  $X_i$ 's are no longer fully observed, though, in order to have versions of the  $F_i$  and  $\tilde{F}_i$ , we will draw inspiration from the general definition of the local variation distribution (Equation 3 in Definition 1). First, define

$$F_{i,d}(t) = \sum_{j \in \mathcal{J}_t} |X_i(t_{j+1}) - X_i(t_j)| \bigg/ \sum_{j=1}^{r-1} |X_i(t_{j+1}) - X_i(t_j)|$$

for  $t \in [0, 1]$  and each  $i = 1, 2, \dots, n$ , where  $\mathcal{J}_t$  is the set of all  $j$ 's satisfying  $t_{j+1} \leq t$ . Note that because we only observe each curve over the grid  $0 \leq t_1 < t_2 < \dots < t_r \leq 1$ , we have replaced the supremum

over all grids in Equation 3 of Definition 1 by just this one (the finest grid we get to observe). Notice that  $F_{i,d}(t) = F_d(t)$  for all  $i = 1, 2, \dots, n$ , where

$$F_d(t) = \sum_{j \in \mathcal{J}_t} |\phi(t_{j+1}) - \phi(t_j)| \bigg/ \sum_{j=1}^{r-1} |\phi(t_{j+1}) - \phi(t_j)|.$$

Clearly,  $F_d$  has jump discontinuities at the grid points  $t_j$ 's, is cadlag, and satisfies  $F_d(t) = 0$  for all  $t \in [0, t_2)$  and  $F_d(t) = 1$  for all  $t \in [t_r, 1]$ . Also, its jumps are at most of size  $a_r = \max_{1 \leq j \leq r-1} |\phi(t_{j+1}) - \phi(t_j)| / \sum_{j=1}^{r-1} |\phi(t_{j+1}) - \phi(t_j)|$ .

For the (discretely) observable warped process, we define

$$\begin{aligned} \tilde{F}_{i,d}(t) &= \sum_{j \in \mathcal{J}_t} |\tilde{X}_i(t_{j+1}) - \tilde{X}_i(t_j)| \bigg/ \sum_{j=1}^{r-1} |\tilde{X}_i(t_{j+1}) - \tilde{X}_i(t_j)| \\ &= \sum_{j \in \mathcal{J}_t} |\phi(s_{i,j+1}) - \phi(s_{i,j})| \bigg/ \sum_{j=1}^{r-1} |\phi(s_{i,j+1}) - \phi(s_{i,j})|, \end{aligned} \quad (5)$$

where  $s_{i,j} = T_i^{-1}(t_j)$  for each  $i$  and  $j$  are unobserved random variables. The  $\tilde{F}_{i,d}$ 's have jump discontinuities at the grid points, and are cadlag. The maximum jump size of  $\tilde{F}_{i,d}$  is  $A_{i,r} = \max_{1 \leq j \leq r-1} |\phi(s_{i,j+1}) - \phi(s_{i,j})| / \sum_{j=1}^{r-1} |\phi(s_{i,j+1}) - \phi(s_{i,j})|$ .

With these definitions in place, we can now adapt Steps 1–5 to the discrete case. In what follows, the generalized inverse of a function  $G$  is denoted by  $G^-$ , i.e.,  $G^-(t) = \inf\{u : G(u) \geq t\}$ . The first two steps will remain invariant, except for the fact that they will now employ the discrete local variation measures. This means that we will not require any tuning parameters or smoothness assumptions to estimate the warp and registration maps. The registration itself (the last three steps) will require some smoothing, of course, if it is to make sense:

Step 1\*: Estimate  $F_d$  by  $\hat{F}_d = \{n^{-1} \sum_{i=1}^n \tilde{F}_{i,d}^-\}^-$ , and  $F_d^-$  by  $\hat{F}_d^* = n^{-1} \sum_{i=1}^n \tilde{F}_{i,d}^-$ .

Step 2\*: Predict the random warp map  $T_i$  by  $\hat{T}_{i,d} = \hat{F}_{i,d}^- \circ \hat{F}_d$  and the registration map  $T_i^{-1}$  by  $\hat{T}_{i,d}^* = \hat{F}_d^* \circ \tilde{F}_{i,d} = \{n^{-1} \sum_{i=1}^n \tilde{F}_{i,d}^-\} \circ \tilde{F}_{i,d}$ .

Step 3\*: Since the  $\tilde{X}_i$ 's are observed discretely, we do not have information about their values between grid points. Thus, we first smooth each of the  $\tilde{X}_{i,d}$  using the Nadaraya-Watson kernel regression estimator for an appropriately chosen kernel  $k$  and bandwidth  $h$ , denoting resulting smoothed functions by  $X_i^\dagger$ ,

$$X_i^\dagger(t) = \sum_{j=1}^r k\left(\frac{t-t_j}{h}\right) \tilde{X}_i(t_j) \bigg/ \sum_{j=1}^r k\left(\frac{t-t_j}{h}\right).$$

Define

$$\hat{X}_i^*(t) = X_i^\dagger(\hat{T}_{i,d}(t)), \quad i = 1, 2, \dots, n$$

to be the registered functional observations and write  $\bar{X}_{r*} = n^{-1} \sum_{i=1}^n \hat{X}_i^*$  for their mean.

Step 4\*: Compute the empirical covariance operator  $\widehat{\mathcal{K}}_{r*}$  of the registered curves  $\hat{X}_i^*$ , and use its leading eigenfunction  $\hat{\phi}_*$  as the estimator of  $\phi$  (again, assume the convention that the sign is correctly identified, i.e.,  $\langle \hat{\phi}_*, \phi \rangle \geq 0$ ).

Step 5\*: Finally, estimate  $\xi_i$  by  $\hat{\xi}_{i*} = \langle \hat{X}_i^*, \hat{\phi}_* \rangle_2$  for each  $i \geq 1$ .



We should point out here that our method is also straightforwardly applicable in the situation where the grid over which the  $\tilde{X}_i$ 's are observed, say,  $0 \leq t_{i,1} < t_{i,2} < \dots < t_{i,r_i} \leq 1$ , differs with  $i$ . The reason for this compatibility is the fact that our approach considers only one curve at a time. We formulate it in the notationally simpler case of a common grid, in order to alleviate the notation in the statement of our asymptotic results in Section 4.

### 3.2.1. Some Practical Issues

As mentioned earlier,  $\tilde{F}_{i,d}$  is a step function with jump discontinuities at the grid points. In particular,  $\tilde{F}_{i,d}(t) = 0$  for  $t \in [0, t_2)$  and  $\tilde{F}_{i,d}(t) = 1$  for  $t \in [t_r, 1]$ . Thus,  $\tilde{F}_{i,d}^-(0) = 0$  and  $\tilde{F}_{i,d}^-(1) = t_r$ , which is less than 1 if  $t_r < 1$ , i.e., the grid does not include the right end-point. In this case,  $\tilde{F}_d(t)$  and thus  $\hat{T}_{i,d}(t)$  is properly defined only for  $t \in [0, t_r]$ . Also,  $\tilde{F}_{i,d}^-(u) \leq t_r$  and equality holds iff  $u \in (\tilde{F}_{i,d}(t_{r-1}), 1]$ . Thus,

$$\begin{aligned} \hat{F}_d(t_r) &= \inf \left\{ u : n^{-1} \sum_{i=1}^n \tilde{F}_{i,d}^-(u) \geq t_r \right\} = \inf \{ u : \tilde{F}_{i,d}^-(u) = t_r \ \forall \ i = 1, 2, \dots, n \} \\ &= \inf \{ u : u \in \cap_{i=1}^n (\tilde{F}_{i,d}(t_{r-1}), 1] \} = \max_{1 \leq i \leq n} \tilde{F}_{i,d}(t_{r-1}). \end{aligned}$$

Then,  $\hat{T}_{i,d}(t_r) = \hat{F}_{i,d}^-(\hat{F}_d(t_r)) = \hat{F}_{i,d}^-(\max_{1 \leq j \leq n} \tilde{F}_{j,d}(t_{r-1})) = t_r$ . One can then extend  $\hat{T}_{i,d}(t)$  to the whole of  $[0, 1]$  by, e.g., linearly interpolating between  $(t_r, \hat{T}_{i,d}(t_r)) = (t_r, t_r)$  and  $(1, 1)$ . This practical modification, in case  $t_r < 1$ , enjoys the same asymptotic properties as the originally defined estimator (Section 4), since the effect of the modification is asymptotically negligible due to the homogeneity assumptions on the grid.

Similarly,  $\hat{F}_d^*(u) = n^{-1} \sum_{i=1}^n \tilde{F}_{i,d}^-(u) = t_r$  iff  $u \in \cap_{i=1}^n (\tilde{F}_{i,d}(t_{r-1}), 1] = (\max_{1 \leq i \leq n} \tilde{F}_{i,d}(t_{r-1}), 1]$ . So, in case  $t_r < 1$ , we have  $\hat{T}_{i,d}^*(1) = \hat{F}_d^*(\tilde{F}_{i,d}(1)) = \hat{F}_d^*(1) = t_r < 1$ . This is not a problem since this estimator is not used in the registration procedure and the problem disappears asymptotically anyway, just as described above.

We conclude this section by noting that, since the estimates  $\hat{T}_{i,d}$  of the warp maps do not involve any smoothing and are obtained from compositions of step functions, the resulting registered curves will not be very smooth. This will be particularly noticeable if the number of grid points is small. Note that even in that case, the estimated mean function will be smoother if the sample size is moderately large. If one is interested in obtaining a smooth registration of the sample curves, the following procedure may be adopted. First, we produce smooth versions of the  $\hat{T}_{i,d}$  by some non-parametric smoothing procedure, e.g., polynomial splines of a fixed degree  $m$ , and call these new estimates as  $\hat{T}_{i,s}$ , say. Then, we plug-in these smoothed estimates of the warp functions and define the new registered observations as  $\hat{X}_i^*(t) := X_i^\dagger(\hat{T}_{i,s}(t))$ . It is well-known that a spline smoothed estimate of a smooth function converges to that function in the  $L_2[0, 1]$  sense provided the oscillations of the function go to zero as the number of knots grows to infinity (see Theorem 6.27 in Schumaker (2007)). The latter holds for the  $\hat{T}_{i,d}$ 's since they lie in  $L_2[0, 1]$  (see equation (2.121) in Theorem 2.59 in Schumaker (2007)). Thus, this modified estimator will also provide consistent registration.

### 3.3. Measurement Error

It can often happen that the discretely observed functional data be additionally contaminated by measurement error. In this case, one has to suitably adapt the registration procedure. In the presence of measurement error, we observe  $Y_{i,d} = \tilde{X}_{i,d} + e_i$ , where  $\tilde{X}_{i,d}$  was defined in Section 3.2, and  $e_i = (\epsilon_{i,1}, \epsilon_{i,2}, \dots, \epsilon_{i,r})'$

with the  $\{\epsilon_{i,j} : j = 1, 2, \dots, r, i = 1, 2, \dots, n\}$  being a collection of i.i.d. error variables with zero mean and variance  $\sigma^2$ , independent of the processes and warp maps. The latent process  $X(\cdot)$  is assumed to be rank one as earlier.

We will modify the registration procedure as follows. First, construct a non-parametric function estimator of  $\tilde{X}'_i$ , which is the derivative of the warped process  $\tilde{X}_i$ , using the observation  $Y_{i,d}$  for each  $i$ , and call this estimator  $\hat{X}_{i,w}^{(1)}(\cdot)$ . Define analogues of the  $\tilde{F}_i$ 's as

$$\tilde{F}_{i,w}(t) = \int_0^t |\hat{X}_{i,w}^{(1)}(u)| du / \int_0^1 |\hat{X}_{i,w}^{(1)}(u)| du, \quad t \in [0, 1].$$

Note that unlike the discrete observation case described in the previous section, we now have fully functional versions of  $\tilde{X}'_i$  for each  $i$ , which allows us to mimic the algorithm in the fully observed scenario in Section 3.1.

Step 1\*\*: Estimate  $F_\phi$  by  $\hat{F}_{\phi,e} = \left(n^{-1} \sum_{i=1}^n \tilde{F}_{i,w}^{-1}\right)^{-1}$ .

Step 2\*\*: Predict the warp map  $T_i$  by  $\hat{T}_{i,e} = \tilde{F}_{i,w}^{-1} \circ \hat{F}_{\phi,e}$ , and the registration map by  $\hat{T}_{i,e}^{-1}$ .

Step 3\*\*: Construct non-parametric function estimators of the  $\tilde{X}_i$ 's using the  $Y_{i,d}$ 's, and call them  $\hat{X}_{i,w}(\cdot)$ 's. Define  $\hat{X}_{i,e}^*(t) = \hat{X}_{i,w}(\hat{T}_{i,e}(t))$ ,  $i = 1, 2, \dots, n$  to be the registered functional observations.

Step 4\*\*: Write  $\bar{X}_{e*} = n^{-1} \sum_{i=1}^n \hat{X}_{i,e}$  for the mean of the registered observations and let  $\widehat{\mathcal{K}}_{e*}$  denote their empirical covariance operator. Take its leading eigenfunction, denoted by  $\hat{\phi}_{e*}$ , as the estimator of  $\phi$  (assuming the same sign convention as earlier).

Step 5\*\*: Finally, estimate  $\xi_i$  by  $\hat{\xi}_{i*,e} = \langle \hat{X}_{i,e}, \hat{\phi}_{e*} \rangle$  for each  $i \geq 1$ .

There are two smoothing steps involved in the above algorithm. Given the large literature on non-parametric smoothing techniques, one can choose any smoother. However, the asymptotic results will depend on the efficiency of the chosen smoothing techniques. From now on in this paper, we will use a local quadratic regression approach with kernel  $k_1(\cdot)$  and bandwidth  $h_1(\cdot)$  for finding  $\hat{X}_{i,w}^{(1)}$ . We will then use a local linear estimator with kernel  $k_2(\cdot)$  and bandwidth  $h_2(\cdot)$  for estimating  $\hat{X}_{i,w}$ . These choices are motivated by the advantages of local polynomial estimators in dealing with boundary effects (see, e.g., [Fan and Gijbels \(1996\)](#) and [Wand and Jones \(1995\)](#) for further details on various smoothing techniques). More details on the choices of smoothing parameters are given in Remark 4 after Theorem 5.

#### 4. Asymptotic Theory

We next study the asymptotic properties of the estimators obtained above. First, in Section 4.1, we develop asymptotic theory assuming that Model 2 has been correctly specified (in other words, that we are in an identifiable regime). We develop separate results for each of the three observation regimes considered (full observation, discrete observation, discrete observation with measurement errors). Then, we consider the effect of model misspecification on the asymptotic performance of our estimators in Section 4.2. In what follows, the space  $C^1[0, 1]$  is equipped with the norm  $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$ , where  $\|\cdot\|_\infty$  is the usual sup-norm. The 2-Wasserstein distance between distributions  $G_1$  and  $G_2$  will be denoted by  $d_W(G_1, G_2)$ .

#### 4.1. Consistency and Weak Convergence

Our first two results concern the fully observed case, as described in Section 3.1. Write  $\mu = E(X_1) = E(\xi_1)\phi$ , and  $\mathcal{K} = COV(X_1) = E(X_1 \otimes X_1) - \mu \otimes \mu$ , where  $(f \otimes g)h = \langle g, h \rangle_2 f$  for any triple  $f, g, h \in L^2[0, 1]$ . Let  $||| \cdot |||$  denote the trace norm for operators on  $L_2[0, 1]$ . The covariance kernel of  $X$  is denoted by  $K(\cdot, \cdot)$  and the empirical covariance kernel of the  $\hat{X}_i$ 's is denoted by  $\hat{K}_r(\cdot, \cdot)$ .

**Theorem 2** (Strong Consistency – Fully Observed Case). *Further to Assumptions 2, assume also that  $\phi'$  is Hölder continuous with exponent  $\alpha \in (0, 1]$ . Then, the estimators in Section 3.2 satisfy the following asymptotic results, where convergence is always with probability one:*

- (a)  $d_W^2(\hat{F}_\phi, F_\phi) \rightarrow 0$  as  $n \rightarrow \infty$ .
- (b)  $|||\hat{T}_i^{-1} - T_i^{-1}|||_\infty \rightarrow 0$  and  $||\hat{T}_i - T_i||_\infty \rightarrow 0$  as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (c)  $||\hat{X}_i - X_i||_\infty \rightarrow 0$  as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (d)  $d_W^2(\hat{F}_i, F_\phi) \rightarrow 0$  as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (e)  $||\bar{X}_r - \mu||_\infty \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\bar{X}_r = n^{-1} \sum_{i=1}^n \hat{X}_i$ .
- (f)  $|||\hat{\mathcal{K}}_r - \mathcal{K}||| \rightarrow 0$  and  $||\hat{K}_r - K||_\infty = \sup_{s,t \in [0,1]} |\hat{K}_r(s,t) - K(s,t)| \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover,  $||\hat{\phi} - \phi||_\infty \rightarrow 0$  and  $|\hat{\xi}_i - \xi_i| \rightarrow 0$  as  $n \rightarrow \infty$  for each  $i \geq 1$ .

Furthermore, if we additionally assume that  $E(|||T'_1|||_\infty) < \infty$  and  $\inf_{t \in [0,1]} T'(t) \geq \delta > 0$  almost surely for a deterministic constant  $\delta$  (call this “Condition 1”), then the following stronger results hold with probability one, in lieu of (b), (c), and (e):

- (b')  $|||\hat{T}_i^{-1} - T_i^{-1}|||_1 \rightarrow 0$  and  $||\hat{T}_i - T_i||_1 \rightarrow 0$  as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (c')  $|||\hat{X}_i - X_i|||_1 \rightarrow 0$  as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (e')  $|||\bar{X}_r - \mu|||_1 \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\bar{X}_r = n^{-1} \sum_{i=1}^n \hat{X}_i$ .

Some remarks are in order:

- Remark 2.** 1. *Uniformity:* It is observed from the proof of the uniform convergence of  $\hat{T}_i^{-1}$  in part (b) of the above theorem that  $\max_{1 \leq i \leq n} |||\hat{T}_i^{-1} - T_i^{-1}|||_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Under Condition 1, the same conclusion is true now with the finer norm  $||| \cdot |||_1$ . The convergence in part (d) also holds uniformly for all  $i = 1, 2, \dots, n$ .
2. *Fisher Consistency:* It can be directly verified that  $\hat{F}_\phi^{-1} = \bar{T} \circ F_\phi^{-1}$  so that  $\hat{F}_\phi = F_\phi \circ \bar{T}^{-1}$ . Also,  $\hat{T}_i = T_i \circ \bar{T}^{-1}$ ,  $\hat{T}_i^{-1} = \bar{T} \circ T_i^{-1}$ , and  $\hat{X}_i = \xi_i \phi \circ \bar{T}^{-1}$  for each  $i$ . Further,  $\hat{\mathcal{K}}_r = n^{-1} \sum_{i=1}^n (\hat{X}_i - \bar{X}_r) \otimes (\hat{X}_i - \bar{X}_r) = \{n^{-1} \sum_{i=1}^n \xi_i^2 - \bar{\xi}^2\} (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1})$ , where  $\bar{\xi} = n^{-1} \sum_{i=1}^n \xi_i$ . Thus,  $\hat{\phi} = (\phi \circ \bar{T}^{-1}) / ||(\phi \circ \bar{T}^{-1})||_2$ , and  $\hat{\xi}_i = \langle \hat{X}_i, \hat{\phi} \rangle = \xi_i ||\phi \circ \bar{T}^{-1}||_2$ . Since all of the above estimators are measurable functions of the sample averages of the  $T_i$ 's, the  $\xi_i$ 's and the  $\xi_i^2$ 's, it follows that all of the above estimators are Fisher consistent for their population counterpart.
3. *An Example:* The condition  $\inf_{t \in [0,1]} T'(t) \geq \delta > 0$  almost surely for a deterministic constant  $\delta$  can be relaxed to  $\inf_{t \in [0,1]} T'(t) \geq \delta_i$  almost surely for i.i.d. positive random variables  $\delta_i$  provided we assume that  $E(\delta_1^{-1}) < \infty$ . An example of random warp functions that satisfy  $\inf_{t \in [0,1]} T'(t) \geq \delta > 0$  can be found Section 8 of Panaretos and Zemel (2016). Define  $\zeta_0(t) = t$  and for  $k \neq 0$ , define  $\zeta_k(t) = t - \sin(\pi kt) / (|k| \pi \beta)$  for some  $\beta > 0$ . If  $K$  is an integer-valued, symmetric random variable, then  $E(\zeta_K) = Id$ . For a fixed  $J \geq 2$ , let  $\{K_j\}_{j=1}^J$  be i.i.d. integer-valued, symmetric random variables, and  $\{U_j\}_{j=1}^{J-1}$  be i.i.d.  $Unif[0, 1]$  random variables independent of the  $K_j$ 's. Define  $T(t) = U_{(1)} \zeta_{K_1}(t) + \sum_{j=1}^{J-1} (U_{(j)} - U_{(j-1)}) \zeta_{K_j}(t) + (1 - U_{(J-1)}) \zeta_{K_J}(t)$ . Then,  $T$  is a strictly

increasing homeomorphism on  $[0, 1]$ ,  $T \in C^1[0, 1]$  surely,  $E(T) = Id$ . Further, it can be easily shown that  $\inf_{t \in [0, 1]} T'(t) \geq 1 - \beta^{-1}$ . Thus, the condition  $\inf_{t \in [0, 1]} T'(t) \geq \delta > 0$  holds if we choose  $\beta = (1 - \delta)^{-1}$ .

Further to strong consistency, we also derive weak convergence of the estimators:

**Theorem 3** (Weak Convergence – Fully Observed Case). *Further to Assumptions 2, assume also that  $\phi'$  is Hölder continuous with exponent  $\alpha \in (0, 1]$ , and that  $E(\|T'_1\|_\infty^2) < \infty$ . Then, the estimators in Section 3.1 satisfy the following asymptotic results,*

- (a)  $nd_W^2(\hat{F}_\phi, F_\phi)$  converges weakly as  $n \rightarrow \infty$ .
- (b)  $\sqrt{n}(\hat{T}_i^{-1} - T_i^{-1})$  and  $\sqrt{n}(\hat{T}_i - T_i)$  converge weakly in the  $C[0, 1]$  topology as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (c)  $\sqrt{n}(\hat{X}_i - X_i)$  converges weakly in the  $C[0, 1]$  topology as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (d)  $nd_W^2(\hat{F}_i, F_\phi)$  converges weakly as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (e)  $\sqrt{n}(\bar{X}_r - \mu)$  converges weakly to a zero mean Gaussian distribution in the  $C[0, 1]$  topology as  $n \rightarrow \infty$ .
- (f)  $\sqrt{n}(\hat{\mathcal{K}}_r - \mathcal{K})$  converges weakly in the topology of Hilbert-Schmidt operators, and  $\sqrt{n}(\hat{K}_r - K)$  converges weakly in the  $C([0, 1]^2)$  topology as  $n \rightarrow \infty$ . In both cases, the limits are zero mean Gaussian distributions. Moreover,  $\sqrt{n}(\hat{\phi} - \phi)$  converges weakly to a zero mean Gaussian distribution in the  $C[0, 1]$  topology, and  $\sqrt{n}(\hat{\xi}_i - \xi_i)$  converges weakly as  $n \rightarrow \infty$  for each  $i \geq 1$ .

Since  $C([0, 1]^k)$  is a stronger topology than  $L_2([0, 1]^k)$  for any finite  $k = 1, 2, \dots$ , it follows that the weak convergence results in the above theorem which hold in the  $C([0, 1]^k)$  topology also hold in the  $L_2([0, 1]^k)$  topology by virtue of the continuous mapping theorem.

We shall now study some the asymptotic properties of the estimators in the discrete observation setup (without measurement error).

**Theorem 4** (Limit Theory – Discretely Observed Case Without Measurement Error). *Further to the conditions of Theorem 3, assume that  $\phi \in C^2[0, 1]$ ,  $\int_0^1 |\phi'(u)|^{-\epsilon} < \infty$  for some  $\epsilon > 0$ , and  $\inf_{t \in [0, 1]} T'(u) \geq \delta > 0$  almost surely for a deterministic constant  $\delta$ . Define  $\alpha = \epsilon/(1 + \epsilon)$ . The kernel  $k(\cdot)$  is assumed to be supported on  $[-1, 1]$ . If  $h = h(n) = o(n^{-1/2})$  and  $r = r(n)$  satisfies  $r \gg n^{1/2\alpha}$  as  $n \rightarrow \infty$ , then the estimators introduced in Section 3.2 satisfy*

- (a)  $d_W^2(\hat{F}_d^*, F_\phi) \rightarrow 0$  as  $n \rightarrow \infty$  almost surely, and  $d_W^2(\hat{F}_d^*, F_\phi) = O_P(n^{-1})$  as  $n \rightarrow \infty$ .
- (b)  $\|\hat{T}_{i,d}^* - T_i^{-1}\|_\infty \rightarrow 0$  and  $\|\hat{T}_{i,d} - T_i\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Further,  $\sqrt{n}(\hat{T}_{i,d}^* - T_i^{-1})$  and  $\sqrt{n}(\hat{T}_{i,d} - T_i)$  converge weakly in the  $L_2[0, 1]$  topology as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (c)  $\|\hat{X}_i^* - X_i\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely, and  $\sqrt{n}(\hat{X}_i^* - X_i)$  converges weakly in the  $L_2[0, 1]$  topology as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (d)  $d_W^2(\hat{F}_i^*, F_\phi) \rightarrow 0$  as  $n \rightarrow \infty$  almost surely, and  $d_W^2(\hat{F}_i^*, F_\phi) = O_P(n^{-1})$  as  $n \rightarrow \infty$  for each  $i \geq 1$ .
- (e)  $\|\bar{X}_{r*} - \mu\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely, and  $\sqrt{n}(\bar{X}_{r*} - \mu)$  converges weakly in the  $L_2[0, 1]$  topology as  $n \rightarrow \infty$ .
- (f)  $\|\hat{\mathcal{K}}_{r*} - \mathcal{K}\| \rightarrow 0$  as  $n \rightarrow \infty$  almost surely, and  $\sqrt{n}(\hat{\mathcal{K}}_{r*} - \mathcal{K})$  converges weakly in the topology of Hilbert-Schmidt operators. Further,  $\|\hat{K}_{r*} - K\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ , and  $\sqrt{n}(\hat{K}_{r*} - K)$  converges weakly in the  $L_2([0, 1]^2)$  topology as  $n \rightarrow \infty$ . Moreover,  $\|\hat{\phi}_* - \phi\|_2 \rightarrow 0$  as  $n \rightarrow \infty$  almost surely, and  $\sqrt{n}(\hat{\phi}_* - \phi)$  converges weakly in the  $L_2[0, 1]$  topology. Also,  $|\hat{\xi}_{i*} - \xi_i| \rightarrow 0$  as  $n \rightarrow \infty$  almost surely, and  $\sqrt{n}(\hat{\xi}_{i*} - \xi_i)$  converges weakly as  $n \rightarrow \infty$  for each  $i \geq 1$ .

In all the weak convergence results stated above, the limits are identical to the corresponding limits obtained in the fully observed scenario in Theorem 3.

- Remark 3.** 1. The asymptotic results remain valid in the case where the grid over which the  $\tilde{X}_i$ 's are observed, say,  $0 \leq t_{i,1} < t_{i,2} < \dots < t_{i,r_i} \leq 1$ , differs with  $i$ . The proof, however, will be notationally quite cumbersome. In this case, the requirement on the grid will be as follows:  $\max_{1 \leq j \leq r_i-1} (t_{j+1} - t_j) = O(r_i^{-1})$  as  $r_i \rightarrow \infty$  for each  $i$ , and  $\tilde{r}_n := \min_{1 \leq i \leq n} r_i$  satisfies  $\tilde{r}_n \gg n^{1/2\alpha}$  as  $n \rightarrow \infty$ .
2. The choice of  $h$  in Theorem 4 is an under-smoothing choice. It is made on account of the absence of measurement errors in the observations, which enables us to under-smooth the data without damaging  $\sqrt{n}$ -consistency. This is unlike what happens in classical non-parametric regression due to the presence of errors in that scenario. Also, the boundary points inflate the bias of the Nadaraya-Watson estimator to an order of  $h$  (the same order as that obtained in Theorem 4 for all points). However, these issues are of no consequence in this scenario. It is also natural to under-smooth in this situation since appropriate under-smoothing retains the features of the curves better and allows estimation at a parametric rate even under non-parametric smoothing. If instead of the Nadaraya-Watson estimator, one uses a local linear estimator with bandwidth  $h$ , then the bias is of order  $h^2$  (even at the boundaries). In this case,  $h$  has to be  $o(n^{-1/4})$  to achieve parametric rates of convergence, which is again an under-smoothing choice. Thus, the choice of smoothing method does not play a crucial role in this setup.
3. Unlike Theorem 3, the weak convergence results are all in the  $L_2$  topology. This is because unlike the fully observed case, the estimators involved are not continuous functions in  $[0, 1]$ . We could not consider the weaker  $D[0, 1]$  topology since not all estimators will be cadlag functions. However, we still retain the strong consistency results in parts (b), (c) and (e) in the sup norm similar to Theorem 2. This is due to the fact that those estimators are uniformly bounded almost surely, and thus have finite sup-norm. Further, in all cases, there is no issue with the measurability of the supremum.
4. The condition  $\phi \in C^2[0, 1]$  can be relaxed to requiring that  $\phi'$  is Lipschitz continuous. Moreover, the requirement  $\int_0^1 |\phi'(u)|^{-\epsilon} < \infty$  for some  $\epsilon > 0$  is not restrictive. Of course, it holds if  $\phi'$  is bounded away from zero on  $[0, 1]$ , in which case one can choose  $\alpha = 1$ . Consider the case when  $\phi \in C^2[0, 1]$  and let  $t_0 \in (0, 1)$  be such that  $\phi'(t_0) = 0$ . If  $\phi''(t_0) > 0$ , then we can choose an interval  $A_\delta = (t_0 - \delta, t_0 + \delta) \subset (0, 1)$  such that  $\inf_{u \in A_\delta} |\phi''(u)| \geq \beta > 0$ . Then, a first order Taylor expansion yields  $\int_{A_\delta} |\phi'(t)|^{-\epsilon} dt \leq \beta^{-\epsilon} \int_{A_\delta} |t - t_0|^{-\epsilon} dt < \infty$  for any  $\epsilon < 1$ . Here, we have used the fact that  $\int_0^\delta t^{-\epsilon} dt < \infty$  for any  $\delta > 0$  iff  $\epsilon < 1$ . Thus, if none of the zeros of  $\phi'$  and  $\phi''$  coincide, then the condition  $\int_0^1 |\phi'(u)|^{-\epsilon} < \infty$  holds for any  $\epsilon < 1$ . In general, if  $\phi \in C^m[0, 1]$  for some  $m \geq 2$ , and  $m'$  be the least integer between 2 and  $m$  such that none of the zeros of  $\phi'$  and  $\phi^{(m')}$  coincide, then  $\int_0^1 |\phi'(u)|^{-\epsilon} < \infty$  holds for any  $\epsilon < 1/(m' - 1)$ .

We finally study the asymptotic properties of the estimators in the modified registration procedure employed when one has contamination by measurement error (described in Section 3.3).

**Theorem 5** (Limit Theory – Measurement Error Case). *In addition to the assumptions of Theorem 3, assume that  $\phi \in C^4[0, 1]$ ,  $\int_0^1 |\phi'(u)|^{-\epsilon} du < \infty$  for some  $\epsilon > 0$ . Define  $\alpha = \epsilon/(1 + \epsilon)$ . Assume that  $T \in C^4[0, 1]$  a.s. and  $\inf_{t \in [0, 1]} T'(u) \geq \delta > 0$  almost surely for a deterministic constant  $\delta$ . The kernels  $k_1(\cdot)$  and  $k_2(\cdot)$  are assumed to be supported on  $[-1, 1]$ , symmetric and continuously differentiable. The errors  $\{\epsilon_{ij}\}$  are assumed to be a.s. bounded. Also assume that  $E\{|\xi_1|^{-2\alpha/(2-\alpha)}\} < \infty$  as well as  $E(\|T_1^{(l)}\|_\infty^2) < \infty$  for  $l = 2, 3, 4$ . The bandwidths satisfy  $h_1, h_2 \rightarrow 0$ ,  $rh_1^3, rh_2 \rightarrow \infty$ . Then, the estimators in Section 3.3*

satisfy the following properties.

- (a)  $d_W^2(\hat{F}_{\phi,e}, F_\phi) = O_P(h_1^{4\alpha} + (rh_1^3)^{-\alpha} + n^{-1})$  as  $n \rightarrow \infty$ .
- (b) Both  $\|\hat{T}_{i,e}^{-1} - T_i^{-1}\|_\infty$  and  $\|\hat{T}_{i,e} - T_i\|_\infty$  are  $O_P(h_1^{2\alpha} + (rh_1^3)^{-\alpha/2} + n^{-1/2})$  as  $n \rightarrow \infty$ .
- (c)  $\|\hat{X}_{i,e}^* - X_i\|_\infty = O_P(h_1^{2\alpha} + (rh_1^3)^{-\alpha/2} + h_2^2 + (rh_2)^{-1/2} + n^{-1/2})$  as  $n \rightarrow \infty$ .
- (d)  $\|\bar{X}_{e*} - \mu\|_\infty = O_P(h_1^{2\alpha} + (rh_1^3)^{-\alpha/2} + h_2^2 + (rh_2)^{-1/2} + n^{-1/2})$  as  $n \rightarrow \infty$ .
- (e)  $\|\|\hat{\mathcal{K}}_{e*} - \mathcal{K}\|\| = O_P(h_1^{2\alpha} + (rh_1^3)^{-\alpha/2} + h_2^2 + (rh_2)^{-1/2} + n^{-1/2})$  as  $n \rightarrow \infty$ . Consequently,  $\|\hat{\phi}_{e*} - \phi\|_2$  and  $|\hat{\xi}_{i*,e} - \xi|$  have the same rates of convergence for each fixed  $i$ .

**Remark 4.** 1. Analogous rates of convergence can also be obtained if one uses different non-parametric smoothing techniques than the ones in the theorem. One may, e.g., use a Nadaraya-Watson estimator in Step 3\*\* with boundary kernels to alleviate the boundary bias problem that is well-known for this estimator (see, e.g., [Wand and Jones \(1995\)](#)). Also, to estimate  $\tilde{X}_i'$ , one may use higher order local polynomials with even orders. However, these will be computationally more intensive as well as need additional smoothness assumptions on the latent process and the warp maps.

2. It is observed in the above theorem that the rates of convergence are slower than the parametric rates achieved in the earlier settings due to the non-parametric smoothing steps involved – especially the estimation of derivatives, which is known to have quite slow rates of convergence. Further, the contributions of the two smoothing steps in the convergence rates are clear. It is well known in local linear regression that the optimal rate for  $h_1$  is  $r^{-1/7}$  and that for  $h_2$  is  $r^{-1/5}$ . With these rates, we have  $d_W^2(\hat{F}_{\phi,e}, F_\phi) = O_P(r^{-4\alpha/7} + n^{-1})$ , and the remaining quantities are  $O_P(r^{-2\alpha/7} + n^{-1/2})$ .

3. Let  $\beta = 2\alpha/(2 - \alpha)$  and observe that  $\beta < 2$  since  $\alpha < 1$ . The condition  $E\{|\xi_1|^{-\beta}\} < \infty$  in Theorem 5 is obviously satisfied if  $|\xi_1|$  is bounded away from zero. Suppose that  $\xi_1$  has a continuous density  $f_\xi$ , say, either on  $[0, \infty)$  or on  $(-\infty, \infty)$  in which case it is assumed to be symmetric about zero. If  $\sup_{y \in [0,a]} f_\xi(y) < \infty$  for some  $a > 0$ , then it is easy to show that  $E\{|\xi_1|^{-\beta}\} < \infty$  if  $\beta < 1 \Leftrightarrow \epsilon < 2$ , which is quite general in view of point (4) in Remark 3. If  $\beta \in [1, 2)$ , then this expectation is finite if  $\sup_{y \in [0,a]} y^{-1} f_\xi(y) < \infty$ .

#### 4.2. Effect of Model Misspecification

In practice, it may happen that the underlying distribution is not exactly of rank one as stipulated by Model 2, and required for identifiability. In this section, we carry out a theoretical analysis of the stability of our procedure to such departures. Since identifiability is lost, it is clear that consistency is no longer achievable. However, we can quantify how much the estimators deviate from their population counterparts, at least asymptotically. Since the model is unidentifiable, strictly speaking there is no unique setting corresponding to the law of the data. For this reason, as a convention, we will assume that a “true” underlying distribution is known and fixed. For simplicity of exposition, we focus on the rank two case. This will be seen to carry the essence of the underlying effects, as we discuss in the last point of Remark 5. To obtain more transparent results, we focus on the case where the underlying functions are completely observable as continuous objects.

Let  $X_i = \xi_{i1}\phi_1 + \xi_{i2}\phi_2$  for  $i = 1, 2, \dots, n$ , where  $\xi_{i1}$  and  $\xi_{i2}$  are uncorrelated. Let  $\mu = E(X_1) = E(\xi_{11})\phi_1 + E(\xi_{12})\phi_2$ . Denote  $\gamma_l^2 = \text{Var}(\xi_{1l})$  and  $Y_{il} = [\xi_{il} - E(\xi_{il})]/\gamma_{il}$  for  $l = 1, 2$ . Then,

$$X_i = \mu + \gamma_1 Y_{i1} \phi_1 + \gamma_2 Y_{i2} \phi_2 \quad (6)$$



gives the Karhunen-Loève expansion of  $X_i$ . The (random) local variation distribution induced by  $X_i$  is  $F_i(t) = \int_0^t |X'_i(u)|du / \int_0^1 |X'_i(u)|du$  for  $t \in [0, 1]$ . Note that contrary to the rank one case, where  $\mu$  did not play a role in  $F_i$  (due to cancellation of the term  $\xi_1$  from the numerator and the denominator), here it cannot be neglected. We will later see that it will play a role in the performance of the estimators. Defining  $\eta = \gamma_2/\gamma_1$ , which is the square root of the inverse of the condition number, it follows that

$$F_i(t) = \frac{\int_0^t |\gamma_1^{-1}\mu'(u) + Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)|du}{\int_0^1 |\gamma_1^{-1}\mu'(u) + Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)|du}.$$

The local variation distribution induced by the observed warped data  $\tilde{X}_i = X_i \circ T_i^{-1}$  is given by

$$\tilde{F}_i(t) = \frac{\int_0^t |\tilde{X}'_i(u)|du}{\int_0^1 |\tilde{X}'_i(u)|du} = \frac{\int_0^{T_i^{-1}(t)} |\gamma_1^{-1}\mu'(u) + Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)|du}{\int_0^1 |\gamma_1^{-1}\mu'(u) + Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)|du} = F_i(T_i^{-1}(t)).$$

The idea is that if under suitable conditions the  $F_i$ 's manifest small variability, then the registration procedure will work quite well. We will illustrate two different situations where this is the case. The estimators of the population parameters will be the same as those considered earlier. The next theorem gives bounds on the estimation errors.

**Theorem 6.** *In the setting of Model 6, define*

$$Z_i = \begin{cases} 2 \int_0^1 |X'_i(u) - \mu'(u)|du / \int_0^1 |X'_i(u)|du & \text{if } \mu' \neq 0, \\ 2\eta \int_0^1 |Y_{i2}\phi'_2(u)|du / \int_0^1 |Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)|du & \text{if } \mu' = 0 \end{cases}, \quad \text{for } i = 1, 2, \dots, n.$$

If  $\mu' \neq 0$ , assume that  $\int_0^1 |\mu'(u)|^{-\epsilon} du < \infty$  for some  $\epsilon > 0$ , and if  $\mu' = 0$ , assume that  $\int_0^1 |\phi'_1(u)|^{-\epsilon} du < \infty$  for some  $\epsilon > 0$ . Set  $\alpha = \epsilon/(1 + \epsilon)$ . Suppose that assumption (M3) holds and that for each  $i = 1, 2$ ,  $\phi_i$  lie in  $C^1[0, 1]$  with the derivative being  $\alpha_i$ -Hölder continuous for some  $\alpha_i \in [0, 1]$ . Also assume that  $E(Z_1^\alpha) < \infty$ . Then:

- (a)  $\limsup_{n \rightarrow \infty} \|\hat{T}_i^{-1} - T_i^{-1}\|_\infty \leq \text{const.} \{E(Z_1^\alpha) + Z_i\}$ , and  $\limsup_{n \rightarrow \infty} \|\hat{T}_i - T_i\|_\infty \leq \text{const.} \|T'_i\|_\infty \{Z_i^\alpha + E^\alpha(Z_1^\alpha)\}$  almost surely, where the constant term is uniform in  $i$ .
- (b)  $\limsup_{n \rightarrow \infty} \|\hat{X}_i - X_i\|_\infty \leq O_P(1) \{E(Z_1^\alpha) + Z_i\}$  almost surely.

**Remark 5.** 1. Theorem 6 reveals that if the  $Z_i$  are small, the effect of misspecification is also small. Here are two such cases:

- (a) When  $\mu' \neq 0$ ,  $Z_i = \int_0^1 |Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)|du / \int_0^1 |\gamma_1^{-1}\mu'(u) + Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)|du$ . So, in this case, if  $|\gamma_1^{-1}\mu'|$  has a large enough contribution compared to  $|Y_{i1}\phi'_1 + \eta Y_{i2}\phi'_2|$  for all  $i$ , then the  $Z_i$ 's are small.
- (b) On the other hand, if  $\mu' = 0$ , then if  $\eta$  is small, i.e., the condition number of the process is large (which essentially implies that the process is “close” to a rank one process provided  $E(\xi_{12}) = 0$ ), then the  $Z_i$ 's are small. This can be compared to the minimum eigenvalue registration principle of Ramsay and Silverman (2005), where one tries to find the warp function that minimises the second eigenvalue of the cross-product matrix between the target function and the registered function. Assume that  $E(\xi_{i1}) = E(\xi_{i2}) = 0$  and without loss of generality that  $\gamma_1 = 1$ . If in reality the true unobserved curves are rank one, i.e., the  $\xi_{i1}\phi_1$  component, and we observe warped versions of the rank two curves  $X_i$ 's, then (in the population case) correct registration is achieved by  $T_i$  if the minimum eigenvalue, namely  $\gamma_2^2 = \eta^2$ , of the expected



cross-product matrix equals zero. Thus, in the empirical case, if  $\eta$  is close to zero, we may expect  $\hat{T}_i$  to be close to  $T_i$  and consequently expect the registration procedure to have good performance.

2. Bounds similar to those in (a) and (b) of Theorem 6 can also be obtained for the mean, the covariance, the  $\gamma_1$ 's and the  $\phi_1$ 's as well as the principal components  $Y_{il}$ 's. We do not include them in the statement of the theorem because they need more complicated conditions involving the parameters.
3. General (possibly infinite) rank situation: Let  $X_i = \mu + \sum_{j=1}^M \gamma_j Y_{ij} \phi_j$  for some  $1 \leq M \leq \infty$ , where the  $\{Y_{ij} : j = 1, 2, \dots, M\}$  are uncorrelated with zero mean and unit variance. Without loss of generality, we assume that  $\gamma_1 > \gamma_2 > \dots \geq 0$ . The errors in estimation when  $\mu' \neq 0$  remain the same as in Theorem 6. When  $\mu' = 0$ , then we define  $Z_i = 2\eta \int_0^1 |Y_{i2}\phi_2'(u) + \sum_{k \geq 3} \delta_k Y_{ik}\phi_k'(u)| du / \int_0^1 |Y_{i1}\phi_1'(u) + \eta[Y_{i2}\phi_2'(u) + \sum_{k \geq 3} \delta_k Y_{ik}\phi_k'(u)]| du$  for  $i = 1, 2, \dots, n$ , where  $\delta_k = \gamma_k/\gamma_2$  for  $k \geq 3$ . In this case, under the conditions of Theorem 6, the bounds as in that theorem still hold true. Note that  $\delta_k \leq 1$  for all  $k \geq 3$ . So, in the general case, the performance of the registration procedure studied in the paper will only depend on how small  $\eta$  is and does not in general depend on the values of the  $\delta_k$ 's (or the  $\gamma_j$ 's for  $j \geq 3$ ). In other words, only the behaviour of the second frequency component relative to the first one matters (which elucidates the role of  $\delta$  in the standard model, i.e. Equation 1, whose role is precisely to tune this behaviour). Of course, the magnitude of the error in estimation for the same value of  $\eta$  will now differ from the rank 2 case because of the presence of the additional terms. We have investigated these issues in a simulation study in Section 5.3 (see, in particular, Figure 7).

## 5. Numerical Experiments

We now carry out some simulation experiments to prove the finite-sample performance of our registration procedure. First we treat the case of a well-specified identifiable model, then the case of a misspecified unidentifiable model, and finally the case when the warped observations under an identifiable model are observed with measurement error.

### 5.1. Well-specified Model without error

Let  $X(t) = \xi\phi(t)$ ,  $t \in [0, 1]$ , and consider two models:

Model 1:  $\xi \sim N(1.5, 1)$ ,  $\phi(t) = 70 - 20 \exp(1.7(t - 0.49)^2) - 50 \exp(-(t - 0.51)^2)$ ;

Model 2:  $\xi \sim 1 + \text{Beta}(2, 2)$ ,  $\phi(t) = \{1 - (t - 0.25)^2\} \cos(3\pi t)$ .

In either case, the sample size is  $n = 50$  and the curves are observed at  $r = 101$  equally spaced points in  $[0, 1]$ . The warp maps are chosen according to point (3) of Remark 2 with the parameters  $J = 2$ ,  $K = V_1 V_2$ , where  $V_1 \sim \text{Poisson}(3)$ ,  $P(V_2 = \pm 1) = 1/2$  with  $V_2$  independent of  $V_1$ , and  $\beta = 1.01$ .

The kernel for the Nadaraya-Watson estimator as well as the one used to smooth the  $\hat{T}_{i,d}$ 's is the Epanechnikov kernel on  $[-1, 1]$ . For both the models, the bandwidths used in the registration procedure were chosen to under-smooth the data so that the features (maxima, minima, etc.) are not smeared out. In order to provide smooth registered curves, we have smoothed the  $\hat{T}_{i,d}$ 's using cubic splines with 11 equi-spaced knots on  $[0, 1]$ , prior to synchronising the data. We present both raw and smoothed registered data curves, means and eigenfunctions.

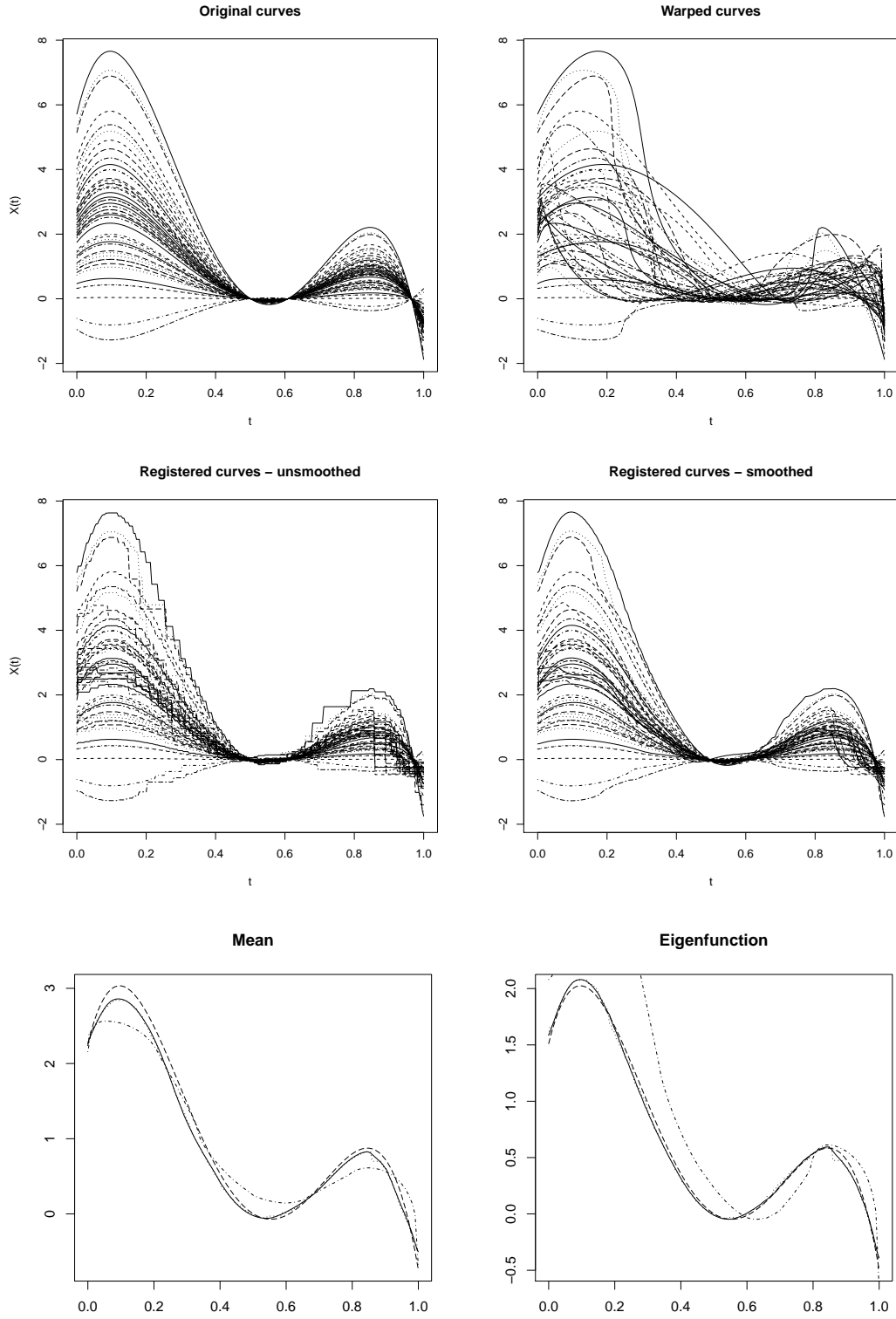


FIG 1. Plots of the true, warped and registered data curves, means and leading eigenfunctions under Model 1. The first plot in the last row shows the true mean (dashed curve), the mean of the un-smoothed registered data (dotted curve), the mean of the smoothed registered data (solid curve), and the mean of the warped data (dot-dashed curve). The second plot in the last row is the analogous plot for the leading eigenfunction.

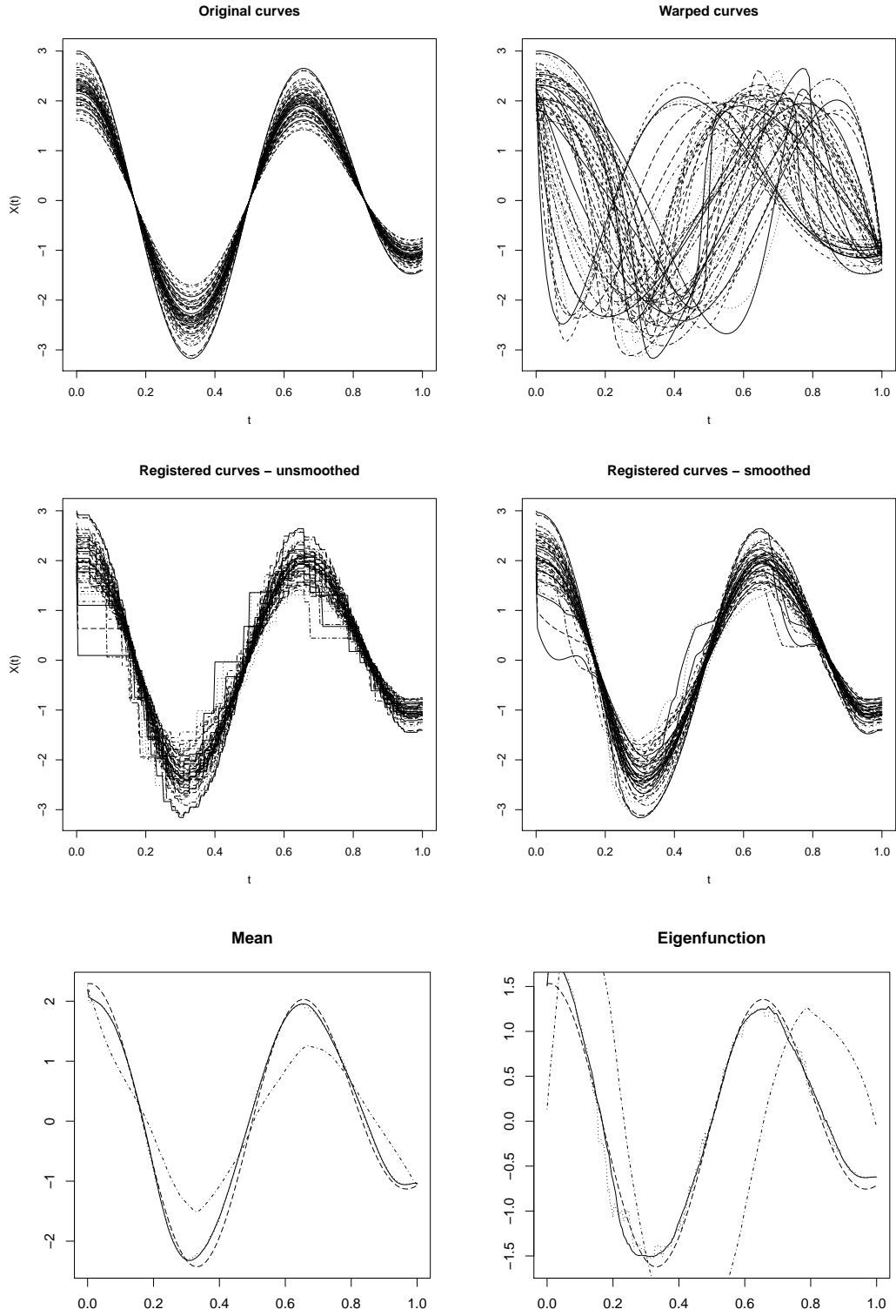


FIG 2. Plots of the true, warped and registered data curves, means and leading eigenfunctions under Model 2. The first plot in the last row shows the true mean (dashed curve), the mean of the un-smoothed registered data (dotted curve), the mean of the smoothed registered data (solid curve), and the mean of the warped data (dot-dashed curve). The second plot in the last row is the analogous plot for the leading eigenfunction.

Figures 1 and 2 show the plots of the true, warped and registered data curves; the true, warped and registered means; and the true, warped and registered leading eigenfunctions under Model 1 and Model 2, respectively. Figures 1 and 2 suggest that the procedure studied in this paper has been able to adequately register the discretely observed and warped sample curves. Moreover, it is clear that the cross-sectional mean and the leading eigenfunction of the warped curves differ from the true mean and leading eigenfunction in either amplitude or phase (under either model), while the registration procedure corrects the problem, and the resulting estimates (whether smoothed or raw) are very close to the true functions.

### 5.2. Well-specified Model with error

We now consider the situation when the warped observations under an identifiable rank one model have been observed with measurement errors. As observed in our theoretical study in Section 4.1, the rate of convergence will be much slower than the case when there is no measurement error. For our simulations, we thus keep the same two models as in Section 5.1 but increase the sample size to  $n = 250$ . The measurement errors in both situations are i.i.d.  $Unif(-0.4, 0.4)$ . The bandwidths for the smoothing steps involved in the registration procedure are chosen using built-in cross-validation methods in the `locpol` package in the R software. Figures 3 and 4 show the plots of the unobserved true rank one curves, the warped curves that are observed with error and the registered curves. They also contain the plots of the mean function and the leading eigenfunction of the true, warped and registered data under the two models. It is observed that even subject to measurement error contamination, the registration procedure is able to adequately register the curves. In particular, under Model 2, the means as well as the leading eigenfunction of the true and the registered curves are quite close. We also performed the registration procedure with a Nadaraya-Watson estimator (without boundary kernels) for obtaining an estimate of the  $\tilde{X}_i$ 's (see Step 3\*\*). Interestingly, the performance was not that different than the one when using a local linear estimator.

### 5.3. Misspecified Model

We next carry out experiments to probe the performance of the registration procedure in a rank 2 and a rank 3 setting – these correspond to an unidentifiable (and hence misspecified) model. The model considered in the rank 2 case are  $X = \xi_1\phi_1 + \xi_2\phi_2$  with  $\xi_1 \sim N(1.5, 1)$ ,  $\xi_2 \sim N(-0.5, 0.15)$ ,  $\phi_1(t) = \sqrt{2}\sin(\pi t)$  and  $\phi_2(t) = \sqrt{2}\cos(2\pi t)$ ,  $t \in [0, 1]$ . In the rank 3 case, we consider  $X = \xi_1\phi_1 + \xi_2\phi_2 + \xi_3\phi_3$  with the same choices of  $\xi_j$  and  $\phi_j$  as above for  $j = 1, 2$  along with  $\xi_3 \sim N(0.5, (0.15)^2)$  and  $\phi_3(t) = \sqrt{2}\cos(4\pi t)$ . The warp maps are the same as those considered in the simulation study in Section 5. The plots of the true curves, the warped curves and the registered curves are provided in Figures 5 and Figure 6 for the rank 2 and the rank 3 models, respectively. It is observed that the registration procedure performs quite well and aligns the peak (present in the true curves) adequately under both models (see Figure 5). Further, the two smaller troughs near the end-points present in the rank 3 model are also reasonably aligned (see Figure 6). There are some amplitude differences between the first two estimated eigenfunctions and the corresponding true eigenfunctions under both models. Surprisingly, the third eigenfunction in the rank 3 model is very well estimated. Overall, even though the rank 1 assumption is violated, the registration procedure performs quite satisfactorily.

In order to probe the breakdown point of the registration procedure in the misspecified setting, we

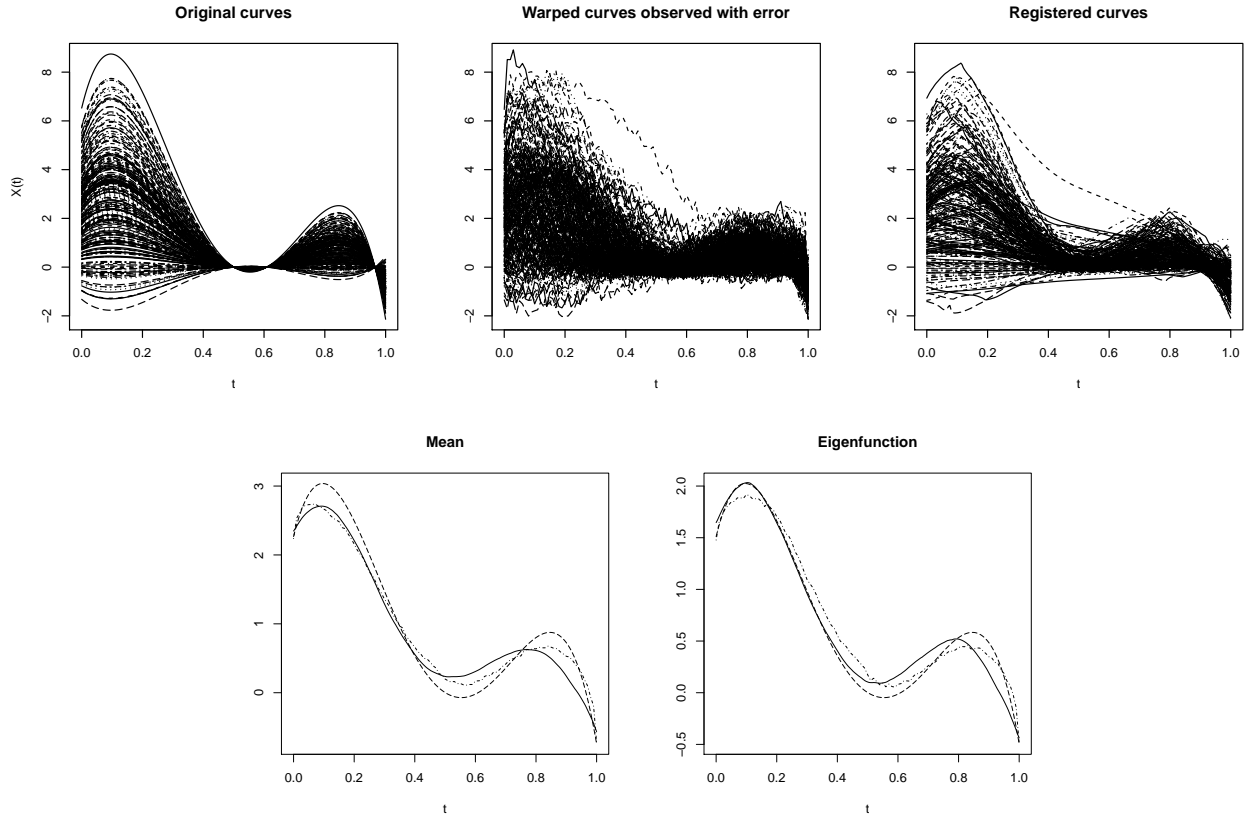


FIG 3. Plots of the true, warped and registered data curves, means and leading eigenfunctions under Model 1 with error. The first plot in the last row shows the true mean (dashed curve), the mean of the registered data (solid curve), and the mean of the warped data (dot-dashed curve). The second plot in the last row is the analogous plot for the leading eigenfunction.

also considered classes of rank 2 and rank 3 models, recorded the relative  $L_2$ -error in estimation of the data curves, i.e, the median of  $\|\hat{X}_i - X_i\|/\|X_i\|, i = 1, 2, \dots, n$ , and consider a threshold of 10% error as a criterion for good performance. The models are generated similar to the earlier simulation. For the rank 2 case, let  $X = \xi_{1,c}\phi_1 + \xi_{2,c,r}\phi_2$ , where  $\xi_1 \sim N(3c, 1)$ ,  $\xi_2 \sim N(-c, r)$ , where  $c \in [0.1, 2]$  and  $r \in [0.01, 0.3]$ . The choices of  $c$  and  $r$  ensure that we include both approximately rank 1 models ( $c$  and  $r$  close to zero) as well as proper rank 2 models (large values of  $r$ ). Similarly, for the rank 3 case, let  $X = \xi_{1,c}\phi_1 + \xi_{2,c,r}\phi_2 + \xi_{3,c,r^2}\phi_3$ , where  $\xi_3 \sim N(c, r^2)$ . Figure 7 shows a plot of the relative  $L_2$ -errors under these classes of models, for various combinations of the parameters  $c$  and  $r$ . It is seen that when  $c$  is large, the performance of the registration procedure is good, which conforms with our theoretical arguments in Theorem 6. In fact, for this class of rank 2 models, the maximum  $L_2$  error does not exceed 12.9%. On the other hand, when  $c$  is small, the allowable range of  $r$  values for good performance is much greater in the rank 2 setup compared to the rank 3 setup (cf. (c) in Remark 5). In fact, in the rank 3 setup, the error is more than 10% for all  $r$  in the range considered when  $c \leq 0.2$ . Further, the maximum  $L_2$  error is now 29.8%.

## 6. Data Analysis

In this section, we illustrate the performance of our registration procedure on a data set of growth curves of *Tribolium* beetle larvae, collected and analysed by Irwin and Carter (2013). Each curve represents the mass measurement (in milligrams) as a function of the age of the larvae since hatching (in days).

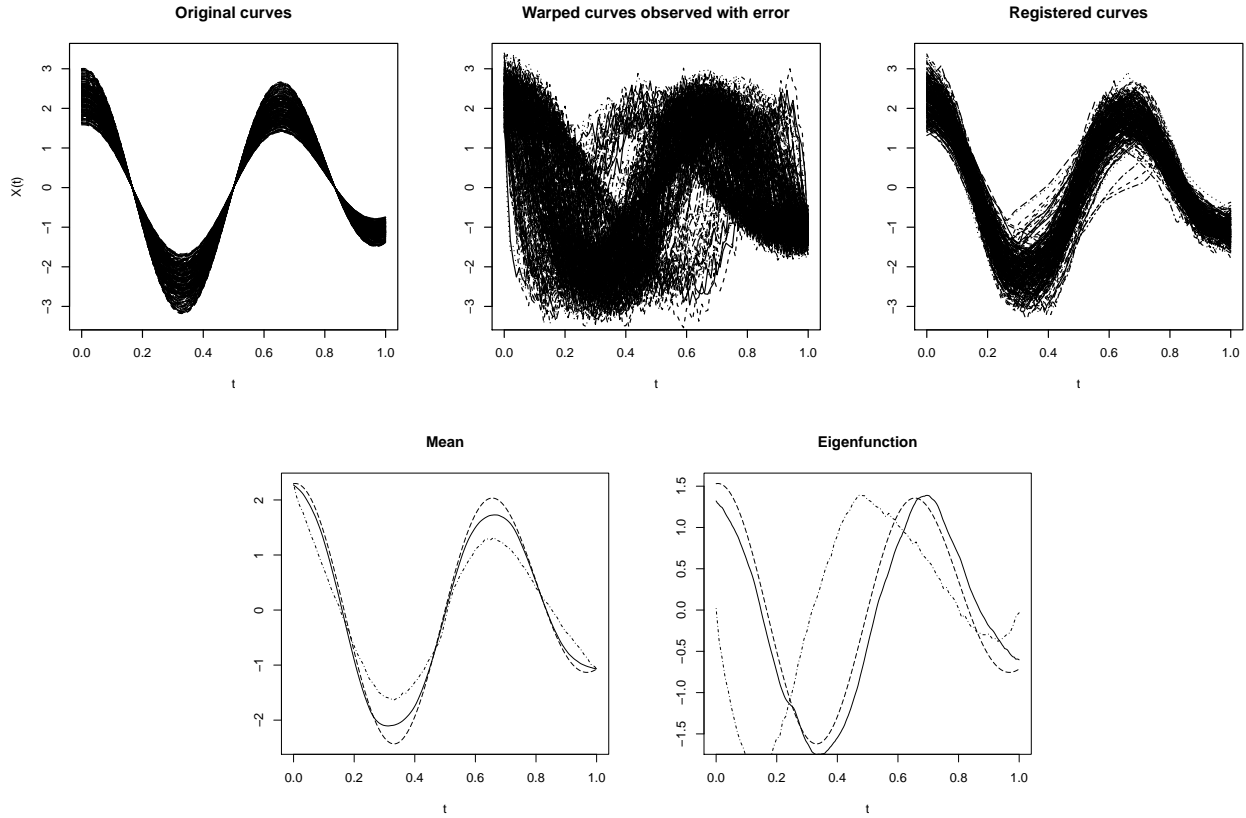


FIG 4. Plots of the true, warped and registered data curves, means and leading eigenfunctions under Model 1 with error. The first plot in the last row shows the true mean (dashed curve), the mean of the registered data (solid curve), and the mean of the warped data (dot-dashed curve). The second plot in the last row is the analogous plot for the leading eigenfunction.

Their analysis of *Tribolium* growth suggests that these beetles' growth patterns differ from those of other animals with determinate growth (that is, growth that is contained in certain life stages). Usually, the longer the growth period, the larger the maximal mass attained (see Irwin and Carter (2014), and references therein). In *Tribolium*, however, it seems that beetles that tend to grow faster, and thus have a shorter growth period, also tend to attain larger size (e.g. Figure 8, top left). See Irwin and Carter (2013) for more details and background. This observation suggests that the *Tribolium* data could be well-suited for a phase-amplitude analysis under a latent rank 1 model that has been warped: one expects that correcting for different “growth clocks” (phase variation) should yield curves that are roughly of unimodal amplitude variation, due to final mass. Conversely, it suggests a potential latent model that produces rank 1 vertical variation related only to final mass, and horizontal variation due to growth timing (i.e. how this total final mass is accumulated in time).

For our analysis, we have only considered the part of the dataset where there were at least 10 discrete measurements per individual curve, which results in a sample size of 159. Also, not all larvae were recorded on the same day so that the number of observations differed across individuals. Since there are relatively few measurements (maximum 12) per individual larvae, we smoothed each observation vector as a pre-processing step. This was done using the built-in function `splinefun` in the R software with the method `monoH.FC` that uses monotone Hermite spline interpolation proposed by Fritsch and Carlson (1980) (since the curves are expected to be approximately increasing).

As is typically the case with growth curves, one expects that, if unaccounted for, the lurking phase

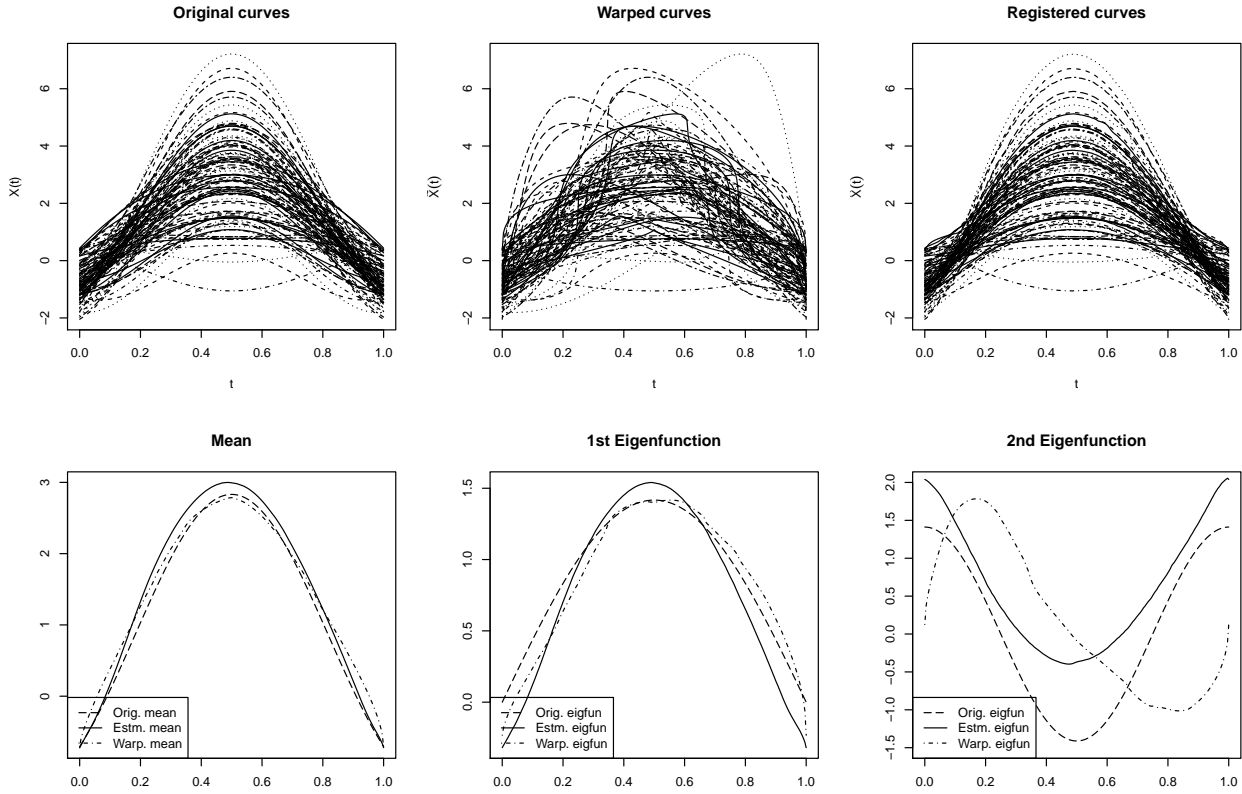


FIG 5. Plots of the true, warped and registered data curves along with the means and eigenfunctions of the true, warped and the registered data under the rank 2 model.

variation would give the impression of several modes of amplitude variation. The aim our analysis is thus to register the curves, estimate the warp maps, estimate the mean of the registered curves, and carry out an eigenanalysis of the registered data.

It is indeed observed that prior to any registration, the data present at least two substantial modes of amplitude variation, with the first three principal components explaining 78.4%, 12% and 3.85% of the total variation, respectively. However, after registration using our method, the empirical covariance operator is almost precisely of rank 1, with the leading principal component explaining 99.72% of the total variation. Interestingly, the mean of the registered data has the same shape as the leading eigenfunction and is in fact roughly equal to 776 times the leading eigenfunction. This can be seen as a model diagnostic, corroborating the model: if the rank 1 model were correct, then after registration one would expect to have a single mode of amplitude variation and a mean in the span of the corresponding eigenfunction (see the discussion after Counterexample 1).

Figure 8 show the plots of the actual data, the monotone spline smoothed data and the registered data, as well as the plot of the estimated warp maps and the average warp map, which is very close to the identity. It also shows the plots of the mean and the leading eigenfunction of the warped and the registered data. Although the means of the warped and the registered data are very close, there are substantial qualitative differences between the corresponding eigenfunctions. The eigenfunction of the registered data shows that the variation in growth pattern essentially starts at about the 8 days after hatching. Between ages 10–16 days post hatching, there is a notable increase in the growth variation, and it somewhat recedes after that age. These periods are in fact compatible with biologically interpretable phases of growth: the larvae enter an “instar” (a distinct growth period between exoskeleton moults) characterised by exponential growth at around day 7-8; then, around day 17, they enter the “wandering



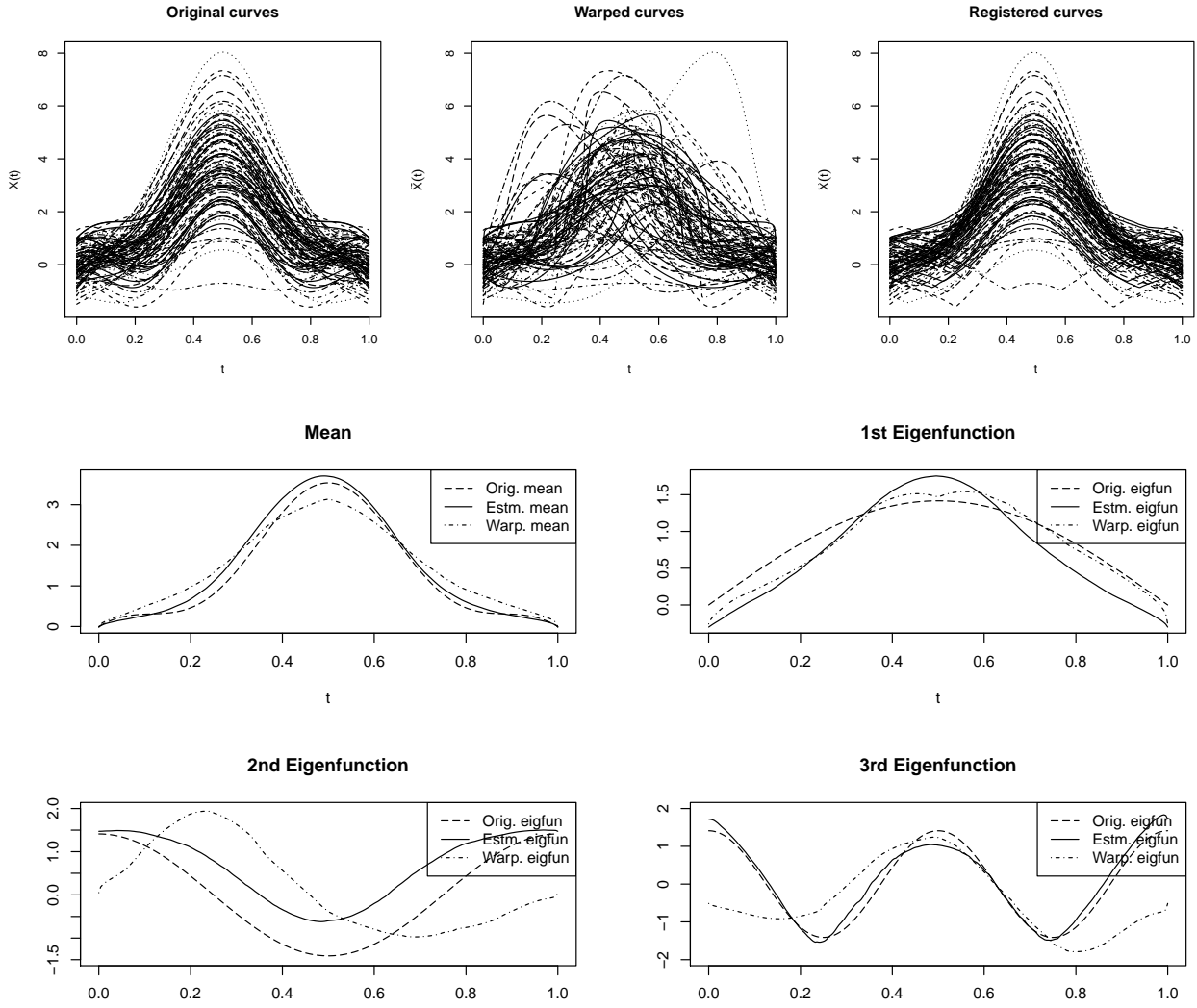


FIG 6. Plots of the true, warped and registered data curves along with the means and eigenfunctions of the true, warped and the registered data under the rank 3 model.

phase” and begin losing weight in preparation for pupation.

## Appendix – Proofs of Formal Statements

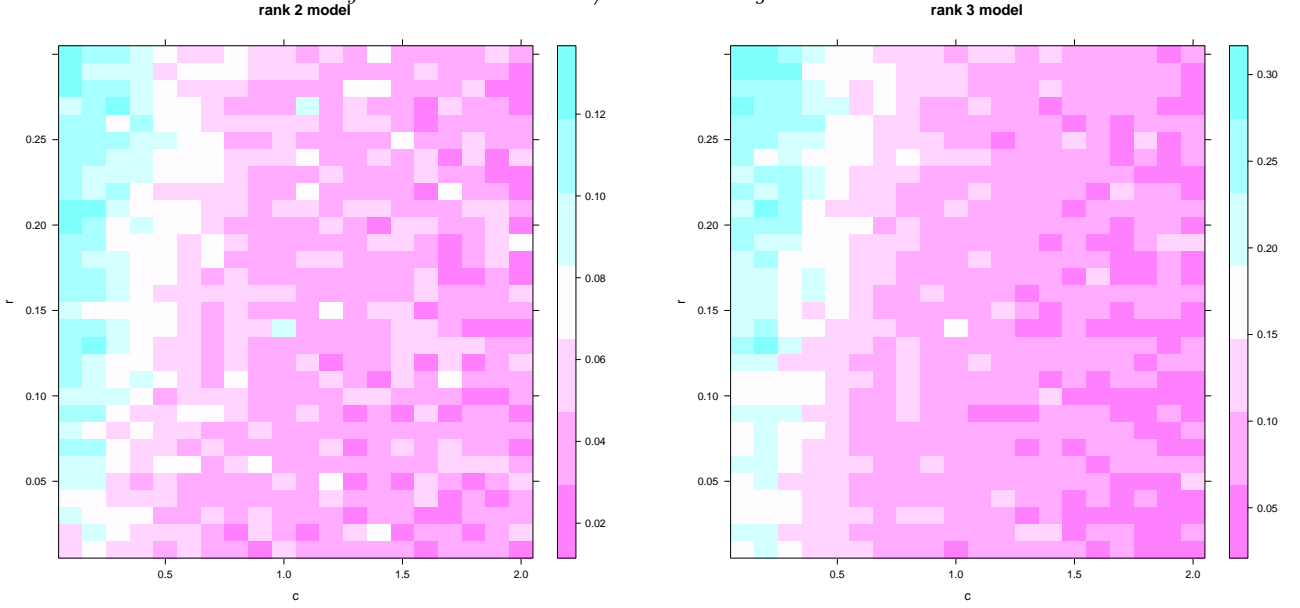
*Proof of Lemma 1.* Since  $X(t) = \xi\phi(t)$ ,  $t \in [0, 1]$ , we have

$$F(t) = \int_0^t |X'(u)|du / \int_0^1 |X'(u)|du = \int_0^t |\phi'(u)|du / \int_0^1 |\phi'(u)|du = F_\phi(t)$$

by Definition 1. Next,  $\tilde{X}(t) = \xi\phi(T^{-1}(t))$  so that  $\tilde{X}'(t) = \xi\phi'(T^{-1}(t))/T'(T^{-1}(t))$ . Thus, using the strict monotonicity of  $T$ , we have

$$\tilde{F}(t) = \int_0^t |\tilde{X}'(u)|du / \int_0^1 |\tilde{X}'(u)|du = \left\{ \int_0^t |\phi'(T^{-1}(u))|/T'(T^{-1}(u))du \right\} / \left\{ \int_0^1 |\phi'(T^{-1}(u))|/T'(T^{-1}(u))du \right\}.$$

A standard change-of-variable argument and the fact that  $T$  is a bijection with  $T(0) = 0$  and  $T(1) = 1$  now yields  $\tilde{F}(t) = \int_0^{T^{-1}(t)} |\phi'(u)|du / \int_0^1 |\phi'(u)|du = F_\phi(T^{-1}(t))$ . So,  $\tilde{F} = F_\phi \circ T^{-1}$ , equivalently,  $T = \tilde{F}^{-1} \circ F_\phi \leftrightarrow T \circ F_\phi^{-1} = \tilde{F}^{-1}$ . Using the assumption that  $E(T) = Id$ , we now have  $E(\tilde{F}^{-1}) = F_\phi^{-1}$ .  $\square$


 FIG 7. Level-plots of the relative  $L_2$  errors under the rank 2 and the rank 3 classes of models.

*Proof of Theorem 1.* Note that  $f : C^1[0, 1] \mapsto f' \in (C[0, 1], \|\cdot\|_\infty)$  is a Lipschitz map. Thus,  $\tilde{X}_1 \stackrel{d}{=} \tilde{X}_2$  implies that  $\tilde{X}'_1 \stackrel{d}{=} \tilde{X}'_2$ . Consider the random probability measure given by

$$\Psi_1(A) = \int_A |\tilde{X}'_1(u)| du / \int_{[0,1]} |\tilde{X}'_1(u)| du$$

for  $A$  in the Borel  $\sigma$ -field of  $[0, 1]$ . Similarly,  $\Psi_2(A) = \int_A |\tilde{X}'_2(u)| du / \int_{[0,1]} |\tilde{X}'_2(u)| du$ . We equip the space  $\mathcal{P}$  of diffuse probability measures on  $[0, 1]$  with the  $L^2$ -Wasserstein metric (see, e.g., Villani (2003)) given by  $d_W(\mu, \nu) = \|F_\nu^{-1} - F_\mu^{-1}\|$ , where  $F_\mu$  and  $F_\nu$  are the distribution functions associated with the probability measures  $\mu$  and  $\nu$ . Now for any  $f_1, f_2 \in C^1[0, 1]$  satisfying  $\int_0^1 |f'_i(u)| du > 0$  for  $i = 1, 2$ , consider the measure  $\mu_i$  with density  $|f'_i(s)| / \int_0^1 |f'_i(u)| du$  for  $i = 1, 2$ . The condition  $\int_0^1 |f'(u)| du > 0$  is equivalent to  $f \neq \text{const.}$ . Using the inequality  $d_W(\mu_1, \mu_2) \leq d_{TV}(\mu_1, \mu_2)$  with the latter being the total variation distance (which holds because  $\mu_1$  and  $\mu_2$  are supported on  $[0, 1]$  – see, e.g., p. 888 in Madras and Sezer (2010)), it follows that

$$\begin{aligned} d_W(\mu_1, \mu_2) &\leq \frac{1}{2} \int_0^1 \left| \frac{|f'_1(s)|}{\int_0^1 |f'_1(u)| du} - \frac{|f'_2(s)|}{\int_0^1 |f'_2(u)| du} \right| ds \\ &\leq \frac{1}{2} \int_0^1 \left| \frac{|f'_1(s)|}{\int_0^1 |f'_1(u)| du} - \frac{|f'_1(s)|}{\int_0^1 |f'_2(u)| du} \right| ds + \frac{1}{2} \int_0^1 \left| \frac{|f'_1(s)|}{\int_0^1 |f'_2(u)| du} - \frac{|f'_2(s)|}{\int_0^1 |f'_2(u)| du} \right| ds \\ &\leq \frac{\int_0^1 |f'_1(s) - f'_2(s)| ds}{\int_0^1 |f'_1(s)| ds} \\ &\leq \frac{\|f'_1 - f'_2\|_\infty}{\int_0^1 |f'_1(s)| ds} \leq \frac{\|f_1 - f_2\|_1}{\int_0^1 |f'_1(s)| ds} \end{aligned}$$

Thus, the embedding  $H : f \mapsto \mu_f$  is continuous when the domain, say,  $\mathcal{A}$  is restricted to the set of all non-constant functions on  $C^1[0, 1]$ . But the set  $\mathcal{A}^c$  is a one dimensional linear subspace spanned by the constant function  $f \equiv 1$ , and this implies that  $\mathcal{A}^c$  is a Borel measurable subset of  $C^1[0, 1]$ . So,  $\mathcal{A}$  is a Borel measurable subset of  $C^1[0, 1]$ . Equip  $\mathcal{A}$  with the Borel  $\sigma$ -field induced from  $C^1[0, 1]$ . Since

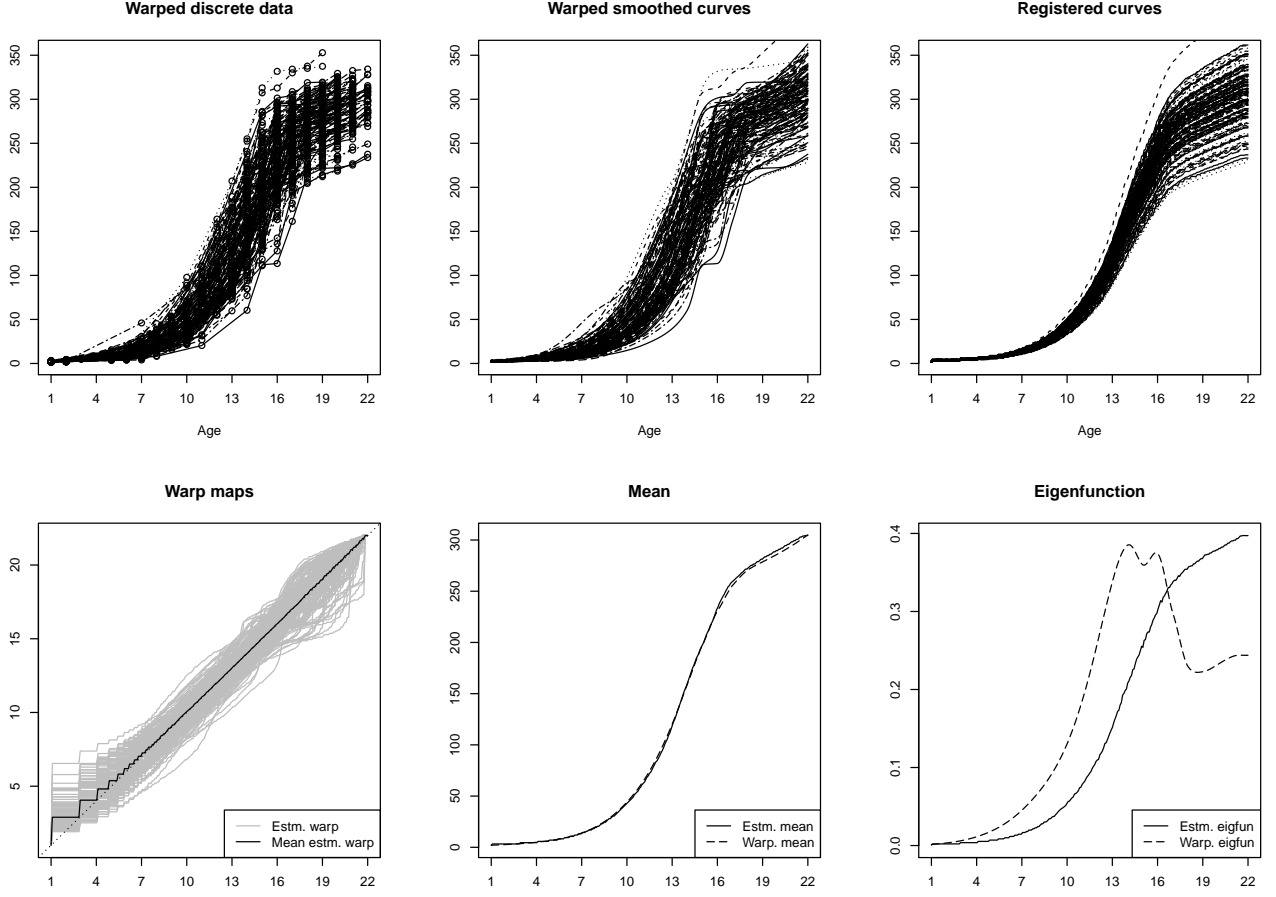


FIG 8. Plots in the first row are those of the Tribolium data, the smoothed curves and the registered curves. The first plot in the second row shows the estimated warp maps, where the dotted line is the identity map. The other two plots in the second row show the means and the leading eigenfunctions of the warped and the registered data.

$P(\tilde{X}_1 \in \mathcal{A}^c) = 0$ , we have that  $H(\tilde{X}_1)$  is a valid random probability measure on  $[0, 1]$ . Note that for any Borel subset  $A$  of  $[0, 1]$ , we have  $H(\tilde{X}_1)(A) = \Psi_1(A)$ . Thus, for any Borel subset  $B$  of  $\mathcal{P}$ , we have

$$P(H(\tilde{X}_1) \in B) = P(\tilde{X}_1 \in H^{-1}(B)) = P(\tilde{X}_2 \in H^{-1}(B)) = P(H(\tilde{X}_2) \in B).$$

The first equality follows from the continuity of  $H$  on  $\mathcal{A}$  and the fact that  $P(\tilde{X}_1 \in \mathcal{A}^c) = 0$  discussed above. The second equality follows from the fact that  $\tilde{X}_1$  and  $\tilde{X}_2$  have the same distributions by assumption. So,  $H(\tilde{X}_1) \stackrel{d}{=} H(\tilde{X}_2)$  as random probability measures.

Next, note that the random measures  $H(\tilde{X}_i)$ ,  $i = 1, 2$ , have strictly increasing cdfs almost surely. Proposition 2 in Panaretos and Zemel (2016) states that for each  $i = 1, 2$ , the map  $\gamma \rightarrow E\{d_W^2(H(\tilde{X}_i), \gamma)\}$  admits a unique minimizer given by  $E\{\tilde{F}_{\Psi_i}^{-1}\}$ , where  $\tilde{F}_{\Psi_i}$  is the random distribution function of the random measure  $H(\tilde{X}_i)$ . Since  $\tilde{X}_i = \xi_i \phi_i(T_i^{-1})$  with  $T_i$  being a strictly increasing homeomorphism on  $[0, 1]$ , it follows from the change-of-variable formula that  $H(\tilde{X}_i)(A) = \Psi_i(A) = \int_{T_i^{-1}(A)} |\phi'_i(u)| du / \int_{[0,1]} |\phi'_i(u)| du$ . Thus,  $\tilde{F}_{\Psi_i} = F_{\phi_i} \circ T_i^{-1}$ , equivalently,  $\tilde{F}_{\Psi_i}^{-1} = T_i \circ F_{\phi_i}^{-1}$ , where  $F_{\phi_i}$  is the cdf associated with the (deterministic) probability measure  $\Phi_i(A) = \int_A |\phi'_i(u)| du / \int_{[0,1]} |\phi'_i(u)| du$ .

Note that  $F_{\phi_i}$  has a continuous and strictly increasing cdf since  $\phi'_i$  is zero only on a countable set for  $i = 1, 2$ . Since  $E(T_i) = Id$ , it follows that the minimizer  $E\{\tilde{F}_{\Psi_i}^{-1}\} = F_{\phi_i}$  for  $i = 1, 2$ . But since  $H(\tilde{X}_1) \stackrel{d}{=} H(\tilde{X}_2)$ , it now follows that  $F_{\phi_1} = F_{\phi_2}$ . Also,  $T_i = \tilde{F}_{\Psi_i}^{-1} \circ F_{\phi_i}$ , equivalently,  $T_i^{-1} = F_{\phi_i}^{-1} \circ \tilde{F}_{\Psi_i}$ . Using the above facts and the result obtained in the previous paragraph, it now follows that  $T_1 \stackrel{d}{=} T_2$ .

We next claim that the joint distributions of  $(\tilde{X}_i, T_i^{-1})$ ,  $i = 1, 2$  are the same. To this end, consider the map  $H_1 : f \mapsto (f, H(f))$  defined from  $\mathcal{A}$  to  $\mathcal{A} \otimes \mathcal{P}$  with the latter being equipped with the induced product topology and the induced product  $\sigma$ -field. It follows from the same arguments used to prove the continuity of  $H$  that  $H_1$  is continuous. Thus, for Borel subsets  $G_1$  and  $G_2$  of  $C^1[0, 1]$ , we have

$$\begin{aligned} P(\tilde{X}_1 \in G_1, T_1^{-1} \in G_2) &= P(\tilde{X}_1 \in G_1, F_{\phi_1}^{-1} \circ \tilde{F}_{\Psi_1} \in G_2) = P(\tilde{X}_1 \in G_1, \tilde{F}_{\Psi_1} \in F_{\phi_1}(G_2)) \\ &= P(H_1(\tilde{X}_1) \in G_1 \times F_{\phi_1}(G_2)) = P(\tilde{X}_1 \in H_1^{-1}(G_1 \times F_{\phi_1}(G_2))) \\ &= P(\tilde{X}_2 \in H_1^{-1}(G_1 \times F_{\phi_2}(G_2))) \quad [\text{since } F_{\phi_1} = F_{\phi_2}] \\ &= P(H_1(\tilde{X}_2) \in G_1 \times F_{\phi_2}(G_2)) = P(\tilde{X}_2 \in G_1, \tilde{F}_{\Psi_2} \in F_{\phi_2}(G_2)) \\ &= P(\tilde{X}_2 \in G_1, F_{\phi_2}^{-1} \circ \tilde{F}_{\Psi_2} \in G_2) = P(\tilde{X}_2 \in G_1, T_2^{-1} \in G_2). \end{aligned}$$

Next, note that  $X_i = \tilde{X}_i \circ T_i$  is the true unobserved process. It is easy to show that the map  $(f, g) \mapsto f \circ g$  from  $C^1[0, 1] \otimes C^1[0, 1]$  into  $C^1[0, 1]$  is continuous. Thus, using the observation in the previous paragraph, we have  $X_1 \stackrel{d}{=} X_2$  as random elements in  $C^1[0, 1]$ . It follows from the equality of distributions that their covariance operators are equal, and thus the corresponding eigenfunctions are equal. Now, the covariance operator of  $X_i$  is given by  $\text{Var}(\xi_i)\phi_i \otimes \phi_i$ . Since  $X_i = \xi_i\phi_i$  is a rank one process, the equality of the covariance operators implies that  $\phi_1 = \pm\phi_2$  (since  $\|\phi_1\|_2 = \|\phi_2\|_2 = 1$ ). This equality along with the fact that  $X_1 \stackrel{d}{=} X_2$  implies that  $\xi_1 = \langle X_1, \phi_1 \rangle_2 \stackrel{d}{=} \langle X_2, \phi_1 \rangle_2 = \langle X_2, \pm\phi_2 \rangle_2 = \pm\xi_2$ .  $\square$

*Proof of Theorem 2.* First observe that the  $T_i$ 's are also i.i.d. random elements in  $C[0, 1]$ . Moreover, since  $T_1$  is strictly increasing and positive, we have  $E(\|T_1\|_\infty) = E(T_1(1)) = 1 < \infty$ . Thus, by the strong law for Banach space valued random elements (see, e.g., Theorem 2.4 in [Bosq \(2000\)](#)), it follows that  $\bar{T} \rightarrow E(T_1) = Id$  as  $n \rightarrow \infty$  almost surely. In addition, if  $E(\|T_1'\|_\infty) < \infty$  implying that  $E(\|T_1\|_1) < \infty$ , then the almost sure convergence  $\bar{T} \rightarrow E(T_1) = Id$  holds in  $C^1[0, 1]$ .

(a) Since  $\hat{F}_\phi^{-1} = \bar{T} \circ F_\phi^{-1}$ , using Theorem 2.18 in [Villani \(2003\)](#), we get that

$$\begin{aligned} d_W^2(\hat{F}_\phi, F_\phi) &= \|\hat{F}_\phi^{-1} - F_\phi^{-1}\|_2^2 \\ &= \int_0^1 \left| \hat{F}_\phi^{-1}(F_\phi(t)) - t \right|^2 F_\phi(dt) \\ &= \int_0^1 |\bar{T}(t) - t|^2 F_\phi dt \leq \|\bar{T} - Id\|_\infty^2 \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

(b) Since each  $T_i$  is a strictly increasing bijection on  $[0, 1]$ , we have

$$\|\hat{T}_i^{-1} - T_i^{-1}\|_\infty = \sup_{t \in [0, 1]} |\bar{T}(T_i^{-1}(t)) - T_i^{-1}(t)| = \|\bar{T} - Id\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since both  $\hat{T}_i^{-1}$  and  $T_i^{-1}$  are strictly increasing homeomorphisms, the uniform convergence of  $\hat{T}_i$  to  $T_i$  follows as a consequence of the above uniform convergence.

Suppose now that Condition 1 holds. We have discussed towards the beginning of the proof that in this case  $\|\bar{T} - Id\|_1 \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. In view of the first half of part (b) of the theorem along with the definition of the  $\|\cdot\|_1$  norm, it is enough to show the uniform convergence of the derivatives. Since each  $T_i$  is a strictly increasing bijection on  $[0, 1]$ , so is  $\bar{T}$  for every  $n \geq 1$ . First note that

$$\|(\hat{T}_i^{-1})' - (T_i^{-1})'\|_\infty = \sup_{t \in [0, 1]} |(\bar{T} \circ T_i^{-1})'(t) - (T_i^{-1})'(t)| = \sup_{t \in [0, 1]} \left| \frac{\bar{T}'(T_i^{-1}(t))}{T_i'(T_i^{-1}(t))} - \frac{1}{T_i'(T_i^{-1}(t))} \right|$$

$$= \sup_{t \in [0,1]} \left| \frac{\bar{T}'(t) - 1}{T'_i(t)} \right| \leq \delta^{-1} \|\bar{T}' - \mathbf{1}\|_\infty,$$

where  $\mathbf{1}$  is the constant function taking value 1. It thus follows from an earlier bound that

$$\|(\hat{T}_i^{-1})' - (T_i^{-1})'\|_1 \leq \|\bar{T} - Id\|_\infty + \delta^{-1} \|\bar{T}' - \mathbf{1}\|_\infty \leq \max(1, \delta^{-1}) \|\bar{T} - Id\|_1 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Next note that  $\bar{T}'(t) = n^{-1} \sum_{i=1}^n T'_i(t) \geq n^{-1} \sum_{i=1}^n \inf_{s \in [0,1]} T'_i(s) = \delta$  so that  $\inf_{t \in [0,1]} \bar{T}'(t) \geq \delta > 0$ . Now,

$$\begin{aligned} \|\hat{T}'_i - T'_i\|_\infty &= \sup_{t \in [0,1]} |(T_i \circ \bar{T}^{-1})'(t) - T'_i(t)| = \sup_{t \in [0,1]} \left| \frac{T'_i(\bar{T}^{-1}(t))}{\bar{T}'(\bar{T}^{-1}(t))} - T'_i(t) \right| = \sup_{t \in [0,1]} \left| \frac{T'_i(t)}{\bar{T}'(t)} - T'_i(\bar{T}(t)) \right| \\ &\leq \sup_{t \in [0,1]} \left| \frac{T'_i(t)}{\bar{T}'(t)} - \frac{T'_i(\bar{T}(t))}{\bar{T}'(\bar{T}(t))} \right| + \sup_{t \in [0,1]} \left| \frac{T'_i(\bar{T}(t))}{\bar{T}'(\bar{T}(t))} - T'_i(\bar{T}(t)) \right| \\ &\leq \delta^{-1} \sup_{t \in [0,1]} |T'_i(t) - T'_i(\bar{T}(t))| + \delta^{-1} \|T'_i\|_\infty \|\bar{T}' - \mathbf{1}\|_\infty. \end{aligned}$$

Since  $T'_i$  is continuous on  $[0, 1]$ , it is uniformly continuous. This and the fact that  $\|\bar{T} - Id\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely implies that  $\sup_{t \in [0,1]} |T'_i(t) - T'_i(\bar{T}(t))| \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Combining this fact with the uniform convergence of  $\bar{T}'$  to  $\mathbf{1}$ , we get that  $\|\hat{T}_i - T_i\|_1 \rightarrow 0$  as  $n \rightarrow \infty$  almost surely.

(c) Note that

$$\|\hat{X}_i - X_i\|_\infty = |\xi_i| \sup_{t \in [0,1]} |\phi(\bar{T}^{-1}(t)) - \phi(t)| = |\xi_i| \sup_{t \in [0,1]} |\phi(\bar{T}(t)) - \phi(t)| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

since  $\|\bar{T} - Id\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely, and  $\phi$  is continuous on  $[0, 1]$  and hence uniformly continuous.

Suppose now that Condition 1 holds. Then, as before,

$$\begin{aligned} \|\hat{X}'_i - X'_i\|_\infty &= |\xi_i| \sup_{t \in [0,1]} \left| \frac{\phi'(\bar{T}^{-1}(t))}{\bar{T}'(\bar{T}^{-1}(t))} - \phi'(t) \right| = |\xi_i| \sup_{t \in [0,1]} \left| \frac{\phi'(t)}{\bar{T}'(t)} - \phi'(\bar{T}(t)) \right| \\ &\leq |\xi_i| \sup_{t \in [0,1]} \left| \frac{\phi'(t)}{\bar{T}'(t)} - \frac{\phi'(\bar{T}(t))}{\bar{T}'(\bar{T}(t))} \right| + |\xi_i| \sup_{t \in [0,1]} \left| \frac{\phi'(\bar{T}(t))}{\bar{T}'(\bar{T}(t))} - \phi'(\bar{T}(t)) \right| \\ &\leq |\xi_i| \delta^{-1} \sup_{t \in [0,1]} |\phi'(t) - \phi'(\bar{T}(t))| + |\xi_i| \|\phi'\|_\infty \delta^{-1} \|\bar{T}' - \mathbf{1}\|_\infty. \end{aligned}$$

Using similar arguments as earlier, we conclude that  $\|\hat{X}'_i - X'_i\|_\infty \rightarrow 0$  and hence  $\|\hat{X}_i - X_i\|_1 \rightarrow 0$  as  $n \rightarrow \infty$  almost surely.

(d) Observe that since  $\hat{X}_i = \xi_i \phi \circ \bar{T}^{-1} = X_i \circ \bar{T}^{-1}$ , it follows from the change-of-variable formula that  $\hat{F}_i = F_\phi \circ \bar{T}^{-1}$ . Thus,

$$d_W^2(\hat{F}_i, F_\phi) = \|\hat{F}_i^{-1} - F_\phi^{-1}\|_2^2 = \|\bar{T} \circ F_\phi^{-1} - F_\phi^{-1}\|_2^2 = \int_0^1 |\bar{T}(t) - t|^2 F_\phi(dt) \leq \|\bar{T} - Id\|_\infty^2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(e) Observe that

$$\|\bar{X}_r - \mu\|_\infty = \|n^{-1} \sum_{i=1}^n (\hat{X}_i - X_i) + n^{-1} \sum_{i=1}^n X_i - \mu\|_\infty \leq n^{-1} \sum_{i=1}^n \|\hat{X}_i - X_i\|_\infty + \|n^{-1} \sum_{i=1}^n X_i - \mu\|_\infty.$$

Since the  $X_i$ 's are i.i.d. random elements in  $C[0, 1]$  with  $E(\|X_1\|_\infty) = E(|\xi_1|)\|\phi\|_\infty < \infty$ , we conclude from the strong law for Banach space valued random elements that  $\|n^{-1} \sum_{i=1}^n X_i - \mu\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Also, from the proof of part (c), we have that

$$n^{-1} \sum_{i=1}^n \|\hat{X}_i - X_i\|_\infty = \sup_{t \in [0,1]} |\phi(\bar{T}(t)) - \phi(t)| \times n^{-1} \sum_{i=1}^n |\xi_i| = \sup_{t \in [0,1]} |\phi(\bar{T}(t)) - \phi(t)| \times \{E(|\xi_1|) + o(1)\}$$

as  $n \rightarrow \infty$  almost surely. Thus, using similar arguments as in part (c) of the theorem, we obtain  $n^{-1} \sum_{i=1}^n \|\hat{X}_i - X_i\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Combining the above facts, we conclude  $\|\bar{X}_r - \mu\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely.

Note that since  $X_i = \xi_i \phi$ , it follows that  $\|n^{-1} \sum_{i=1}^n X'_i - \mu'\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Now, suppose that Condition 1 holds. A similar decomposition as above yields

$$\|\bar{X}'_r - \mu'\|_\infty \leq n^{-1} \sum_{i=1}^n \|\hat{X}'_i - X'_i\|_\infty + \|n^{-1} \sum_{i=1}^n X'_i - \mu'\|_\infty.$$

The proof of part (c) implies that

$$n^{-1} \sum_{i=1}^n \|\hat{X}'_i - X'_i\|_\infty \leq \delta^{-1} \left( n^{-1} \sum_{i=1}^n |\xi_i| \right) \left\{ \sup_{t \in [0,1]} |\phi'(t) - \phi'(\bar{T}'(t))| + \|\phi'\|_\infty \|\bar{T}' - \mathbf{1}\|_\infty \right\}.$$

The right-hand term above converges to zero as  $n \rightarrow \infty$  almost surely. The result is now established upon combining the above facts.

(f) Straightforward algebraic manipulations yield

$$\begin{aligned} \widehat{\mathcal{K}}_r &= n^{-1} \sum_{i=1}^n (\hat{X}_i - \bar{X}_r) \otimes (\hat{X}_i - \bar{X}_r) \\ &= n^{-1} \sum_{i=1}^n (X_i - \bar{X}) \otimes (X_i - \bar{X}) + n^{-1} \sum_{i=1}^n (\hat{X}_i - X_i) \otimes (\hat{X}_i - X_i) - (\bar{X} - \bar{X}_r) \otimes (\bar{X} - \bar{X}_r) \\ &\quad + n^{-1} \sum_{i=1}^n \{(\hat{X}_i - X_i) \otimes (X_i - \bar{X}) + (X_i - \bar{X}) \otimes (\hat{X}_i - X_i)\}. \end{aligned}$$

Denote  $\widehat{\mathcal{K}} = n^{-1} \sum_{i=1}^n (X_i - \bar{X}) \otimes (X_i - \bar{X})$ . Then,

$$\|\widehat{\mathcal{K}}_r - \widehat{\mathcal{K}}\| \leq \frac{2}{n} \sum_{i=1}^n \|\hat{X}_i - X_i\|_2 \|X_i - \bar{X}\|_2 + \frac{1}{n} \sum_{i=1}^n \|\hat{X}_i - X_i\|_2^2 + \|\bar{X} - \bar{X}_r\|_2^2.$$

Using the Cauchy-Schwarz inequality, we have  $n^{-1} \sum_{i=1}^n \|\hat{X}_i - X_i\|_2 \|X_i - \bar{X}\|_2 \leq \{n^{-1} \sum_{i=1}^n \|\hat{X}_i - X_i\|_2^2\}^{1/2} \{n^{-1} \sum_{i=1}^n \|X_i - \bar{X}\|_2^2\}^{1/2}$ , and  $n^{-1} \sum_{i=1}^n \|X_i - \bar{X}\|_2^2 = O(1)$  as  $n \rightarrow \infty$  almost surely. It follows from the arguments in the proof of part (c) of the theorem that

$$n^{-1} \sum_{i=1}^n \|\hat{X}_i - X_i\|_2^2 \leq n^{-1} \sum_{i=1}^n \|\hat{X}_i - X_i\|_\infty^2 \leq \sup_{t \in [0,1]} |\phi(\bar{T}(t)) - \phi(t)|^2 \left( n^{-1} \sum_{i=1}^n |\xi_i|^2 \right),$$

and the right hand side is  $o(1)$  as  $n \rightarrow \infty$  almost surely since  $E(|\xi_1|^2) < \infty$ . Further,  $\|\bar{X} - \bar{X}_r\|_2^2 = o(1)$  as  $n \rightarrow \infty$  almost surely. Thus,  $\|\widehat{\mathcal{K}}_r - \widehat{\mathcal{K}}\| = o(1)$  as  $n \rightarrow \infty$  almost surely.

The proof of the uniform convergence of  $\hat{K}_r(s, t)$  to  $K(s, t)$  is obtained by use of a decomposition of  $\hat{K}_r(s, t)$  similar to the one used above, noting that  $\hat{K}(s, t)$  converges uniformly to  $K(s, t)$  (by the strong

law of large numbers in  $C([0, 1]^2)$ , and the fact that all the other bounds hold in the supremum norm.

Next, note that  $\hat{\phi}(t) = \hat{\lambda}^{-1} \int_0^1 \hat{K}_r(s, t) \hat{\phi}(s) ds$  and  $\phi(t) = \lambda^{-1} \int_0^1 K(s, t) \phi(s) ds$  for all  $t \in [0, 1]$ , where  $|\hat{\lambda} - \lambda| \leq |||\hat{\mathcal{K}}_r - \mathcal{K}||| \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Also,  $\|\hat{\phi} - \phi\|_2 \leq 2\sqrt{2}\lambda^{-1} |||\hat{\mathcal{K}}_r - \mathcal{K}||| \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. So,

$$\begin{aligned} |\hat{\phi}(t) - \phi(t)| &\leq \left| \hat{\lambda}^{-1} \int_0^1 \hat{K}_r(s, t) \hat{\phi}(s) ds - \hat{\lambda}^{-1} \int_0^1 K(s, t) \hat{\phi}(s) ds \right| \\ &\quad + \left| \hat{\lambda}^{-1} \int_0^1 K(s, t) \hat{\phi}(s) ds - \hat{\lambda}^{-1} \int_0^1 K(s, t) \phi(s) ds \right| \\ &\quad + \left| \hat{\lambda}^{-1} \int_0^1 K(s, t) \phi(s) ds - \lambda^{-1} \int_0^1 K(s, t) \phi(s) ds \right| \\ &\leq \hat{\lambda}^{-1} \|\hat{K}_r - K\|_\infty + \hat{\lambda}^{-1} \|K\|_\infty \|\hat{\phi} - \phi\|_2 + |(\hat{\lambda}^{-1} - \lambda^{-1})\lambda\phi(t)| \\ &\leq (\lambda^{-1} + o(1)) \{ \|\hat{K}_r - K\|_\infty + \|K\|_\infty \|\hat{\phi} - \phi\|_2 \} + |\lambda - \hat{\lambda}|(\lambda^{-1} + o(1))^{-1} \|\phi\|_\infty \end{aligned}$$

as  $n \rightarrow \infty$  almost surely. Thus,  $\|\hat{\phi} - \phi\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely.

Finally,  $|\hat{\xi}_i - \xi_i| = |\langle \hat{X}_i, \hat{\phi} \rangle - \langle X_i, \phi \rangle| \leq |\langle \hat{X}_i - X_i, \hat{\phi} \rangle| + |\langle X_i, \hat{\phi} - \phi \rangle| \leq \|\hat{X}_i - X_i\|_\infty + \|\hat{\phi} - \phi\|_2 \rightarrow 0$  as  $n \rightarrow \infty$  almost surely.  $\square$

*Proof of Theorem 3.* We have  $|T_1(t) - T_1(s)| \leq \|T_1'\|_\infty |s - t|$  and by assumption  $E(\|T_1'\|_\infty^2) < \infty$ . So, by the CLT for i.i.d.  $C[0, 1]$  valued random elements (see, e.g., Theorem 2.4 Bosq (2000)), we have  $\sqrt{n}(\bar{T} - Id) \xrightarrow{d} Y$  for a zero mean Gaussian random element  $Y$  in  $C[0, 1]$ .

(a) From the proof of part (a) of Theorem 2, one has that  $d_W^2(\hat{F}_\phi, F_\phi) = \int_0^1 |\bar{T}(t) - t|^2 F_\phi(dt)$ . Now, it is easy to check that the map  $C[0, 1] \ni f \rightarrow \int_0^1 |f(t)|^2 F_\phi(dt)$  is continuous. The result follows from the continuous mapping theorem.

(b) Note that for each fixed  $i \geq 1$ , we have  $\sqrt{n}(\hat{T}_i^{-1} - T_i^{-1}) = U_n \circ V_n$ , where  $U_n = \sqrt{n}(\bar{T} - Id)$  and  $V_n = T_i^{-1}$ . We will first derive the weak limit conditional on  $T_i = t_i$ . From the previous paragraph, it follows that conditional on  $T_i = t_i$ ,  $U_n = \sqrt{n}(n^{-1}t_i + n^{-1} \sum_{j \neq i} T_j - Id) \xrightarrow{d} Y$ , and  $V_n$ , being a constant sequence, converges conditionally in probability to  $t_i^{-1}$  as  $n \rightarrow \infty$ . So, by Theorem 4.4 in Billingsley (1968), conditional on  $T_i = t_i$ , we have  $(U_n, V_n) \xrightarrow{d} (Y, t_i^{-1})$  in the  $C[0, 1]$  topology. Using the fact that the map  $(f, g) \mapsto f \circ g$  is continuous in  $C([0, 1]^2)$  (see, e.g., p. 155 in Billingsley (1968)), it follows from the continuous mapping theorem that conditional on  $T_i = t_i$ ,  $\sqrt{n}(\hat{T}_i^{-1} - T_i^{-1}) \xrightarrow{d} Y \circ t_i^{-1}$  as  $n \rightarrow \infty$  for each fixed  $i \geq 1$ . Thus, by the Dominated Convergence Theorem, the unconditional distribution of  $\sqrt{n}(\hat{T}_i^{-1} - T_i^{-1})$  converges weakly as  $n \rightarrow \infty$  for each fixed  $i \geq 1$ .

To prove the weak convergence of  $\sqrt{n}(\hat{T}_i - T_i) = \sqrt{n}(T_i \circ \bar{T}^{-1} - T_i)$ , we will as earlier first derive its weak limit conditional on  $T_i = t_i$ . Now, using the fact that  $T_i' \in C[0, 1]$  almost surely, we have

$$\begin{aligned} \hat{T}_i(s) - t_i(s) &= t_i(\bar{T}^{-1}(s)) - t_i(s) = t_i(s + \bar{T}^{-1}(s) - s) - t_i(s) \\ &= (\bar{T}^{-1}(s) - s) \times t_i'(s + \beta(\bar{T}^{-1}(s) - s)) \end{aligned}$$

for some  $\beta_1 \in [0, 1]$  (possibly depending on  $s$  and  $i$ ). Thus,

$$\sqrt{n}(\hat{T}_i - t_i) = \{\sqrt{n}(\bar{T}^{-1} - Id)\} \times t_i'(\cdot + o_P(1)) = \{\sqrt{n}(Id - \bar{T}) \circ \bar{T}^{-1}\} \times t_i'(\cdot + o_P(1))$$

where the  $o_P(1)$  term is uniform in  $s$  since  $\|\bar{T}^{-1} - Id\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Using similar arguments as in the above proof and noting that  $\|\bar{T} - Id\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely, we deduce that  $\sqrt{n}(\hat{T}_i - t_i) \xrightarrow{d} Y \times t_i'$  as  $n \rightarrow \infty$ . Thus, by the Dominated Convergence Theorem, the unconditional



distribution of  $\sqrt{n}(\hat{T}_i - T_i)$  converges weakly as  $n \rightarrow \infty$  for each fixed  $i \geq 1$ .

(c) Note that for each fixed  $i \geq 1$ ,

$$\begin{aligned}\hat{X}_i(s) - X_i(s) &= \xi_i\{\phi(\bar{T}^{-1}(s)) - \phi(s)\} = \xi_i\{(\bar{T}^{-1}(s) - s)\phi'(s + \beta_2(\bar{T}^{-1}(s) - s))\} \\ \Rightarrow \quad \sqrt{n}(\hat{X}_i - X_i) &= \xi_i\{\sqrt{n}(Id - \bar{T}) \circ \bar{T}^{-1}\} \times \phi'(\cdot + o_P(1)),\end{aligned}$$

where  $\beta_2 \in [0, 1]$ , and the  $o_P(1)$  term is uniform in  $s$  as earlier. Similar arguments as in part (b) above yield  $\sqrt{n}(\hat{X}_i - X_i) \xrightarrow{d} \xi_i Y \times \phi'$  as  $n \rightarrow \infty$  for each fixed  $i \geq 1$ .

(d) The proof is similar to that of part (a) and is omitted.

(e) Note that

$$\begin{aligned}\sqrt{n}(\bar{X}_r - \mu) &= \sqrt{n} \left\{ n^{-1} \sum_{i=1}^n \xi_i \phi \circ \bar{T}^{-1} - E(\xi_1) \phi \right\} \\ &= \sqrt{n} \left\{ n^{-1} \sum_{i=1}^n (\xi_i - E(\xi_1)) \right\} \phi \circ \bar{T}^{-1} + E(\xi_1) \sqrt{n} \left\{ \phi \circ \bar{T}^{-1} - \phi \right\} \\ &\xrightarrow{d} N(0, Var(\xi_1)) \phi + E(\xi_1) Y \times \phi',\end{aligned}$$

which follows from similar arguments as in part (c) and the independence of the  $\xi_i$ 's and the  $T_i$ 's.

(f) For the first part, note that

$$\begin{aligned}\widehat{\mathcal{K}}_r &= n^{-1} \sum_{i=1}^n (\hat{X}_i - \bar{X}_r) \otimes (\hat{X}_i - \bar{X}_r) \\ &= n^{-1} \sum_{i=1}^n (\hat{X}_i - \mu) \otimes (\hat{X}_i - \mu) - (\bar{X}_r - \mu) \otimes (\bar{X}_r - \mu) \\ &= S_1 + S_2, \quad \text{say.}\end{aligned}$$

Now, some straightforward manipulations yield

$$\begin{aligned}S_1 &= n^{-1} \sum_{i=1}^n \{\xi_i \phi \circ \bar{T}^{-1} - E(\xi_1) \phi\} \otimes \{\xi_i \phi \circ \bar{T}^{-1} - E(\xi_1) \phi\} \\ &= n^{-1} \sum_{i=1}^n \{\xi_i - E(\xi_1)\}^2 (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1}) + E^2(\xi_1) (\phi \circ \bar{T}^{-1} - \phi) \otimes (\phi \circ \bar{T}^{-1} - \phi) \\ &\quad + n^{-1} E(\xi_1) \sum_{i=1}^n \{\xi_i - E(\xi_1)\} \left[ (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1} - \phi) + (\phi \circ \bar{T}^{-1} - \phi) \otimes (\phi \circ \bar{T}^{-1}) \right].\end{aligned}$$

So,

$$\begin{aligned}&\sqrt{n}(S_1 - \mathcal{K}) \\ &= \sqrt{n} \left\{ n^{-1} \sum_{i=1}^n \{\xi_i - E(\xi_1)\}^2 (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1}) - \mathcal{K} \right\} \\ &= \sqrt{n} \left\{ n^{-1} \sum_{i=1}^n \{\xi_i - E(\xi_1)\}^2 (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1}) - Var(\xi_1) \phi \otimes \phi \right\} \\ &= \sqrt{n} \left\{ n^{-1} \sum_{i=1}^n [\{\xi_i - E(\xi_1)\}^2 - Var(\xi_1)] (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1}) \right. \\ &\quad \left. + Var(\xi_1) [(\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1}) - \phi \otimes \phi] \right\}\end{aligned}$$

$$\begin{aligned}
& + E^2(\xi_1)(\phi \circ \bar{T}^{-1} - \phi) \otimes (\phi \circ \bar{T}^{-1} - \phi) \\
& + n^{-1}E(\xi_1) \sum_{i=1}^n \{\xi_i - E(\xi_1)\} \left[ (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1} - \phi) + (\phi \circ \bar{T}^{-1} - \phi) \otimes (\phi \circ \bar{T}^{-1}) \right] \Big\}
\end{aligned}$$

The first term on the right hand side of the above equality converges in distribution to  $N(0, E\{\xi_1 - E(\xi_1)\}^4)\phi \otimes \phi$  since  $\bar{T} \rightarrow Id$  as  $n \rightarrow \infty$  almost surely. For the latter reason, the third and the fourth terms converge to zero in probability as  $n \rightarrow \infty$ . For the second term, note that

$$\begin{aligned}
& (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1}) - \phi \otimes \phi \\
& = (\phi \circ \bar{T}^{-1} - \phi) \otimes \phi + (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1} - \phi).
\end{aligned}$$

Thus, by similar arguments as in part (c) earlier, and the continuity of the mapping  $(f, g) \mapsto f \otimes g$  from  $L_2([0, 1]^2)$  to the space of Hilbert Schmidt operators, we have that the second term converges in distribution to  $Var(\xi_1)\{(Y \times \phi') \otimes \phi + \phi \otimes (Y \times \phi')\}$ . Combining the above observations and the fact that  $\sqrt{n}S_2 \rightarrow 0$  in probability (follows from part (e)), we deduce that

$$\sqrt{n}(\widehat{\mathcal{K}}_r - \mathcal{K}) \xrightarrow{d} N(0, E\{\xi_1 - E(\xi_1)\}^4)\phi \otimes \phi + Var(\xi_1)\{(Y \times \phi') \otimes \phi + \phi \otimes (Y \times \phi')\}$$

as  $n \rightarrow \infty$ .

In order to prove the weak convergence of the empirical process  $\{\sqrt{n}(\widehat{K}_r(s, t) - K(s, t)) : s, t \in [0, 1]\}$  in  $C([0, 1]^2)$ , we follow the same decomposition as in the proof of the weak convergence of the operators in the Hilbert Schmidt topology. Now, note that the proof of part (c) of the theorem implies that the empirical process  $\{\sqrt{n}(\phi(\bar{T}^{-1}(t)) - \phi(t)) : t \in [0, 1]\}$  in  $C[0, 1]$  converges in distribution to the process  $\{Y(t)\phi'(t) : t \in [0, 1]\}$  in  $C[0, 1]$ . This fact and the same arguments as in part (f) yield

$$\begin{aligned}
& \{\sqrt{n}(\widehat{K}_r(s, t) - K(s, t)) : s, t \in [0, 1]\} \\
& \xrightarrow{d} \{Z\phi(s)\phi(t) + Var(\xi_1)[Y(s)\phi'(s)\phi(t) + Y(t)\phi'(t)\phi(s)] : s, t \in [0, 1]\}
\end{aligned}$$

as  $n \rightarrow \infty$ , where  $Z \sim N(0, E\{\xi_1 - E(\xi_1)\}^4)$  does not depend on  $s, t$ .

For the weak convergence of  $\widehat{\phi}$ , first note that  $\widehat{\mathcal{K}}_r = n^{-1} \sum_{i=1}^n (\xi_i - \bar{\xi})^2 (\phi \circ \bar{T}^{-1}) \otimes (\phi \circ \bar{T}^{-1})$ . Thus,  $\widehat{\phi} = (\phi \circ \bar{T}^{-1}) / \|\phi \circ \bar{T}^{-1}\|_2$ . Now,

$$\begin{aligned}
\widehat{\phi} - \phi & = \frac{\phi \circ \bar{T}^{-1}}{\|\phi \circ \bar{T}^{-1}\|_2} - \phi = \frac{\phi \circ \bar{T}^{-1} - \phi}{\|\phi \circ \bar{T}^{-1}\|_2} - \frac{\phi(\|\phi \circ \bar{T}^{-1}\|_2 - 1)}{\|\phi \circ \bar{T}^{-1}\|_2} \\
& = \frac{\phi \circ \bar{T}^{-1} - \phi}{\|\phi \circ \bar{T}^{-1}\|_2} - \frac{\phi(\|\phi \circ \bar{T}^{-1}\|_2^2 - 1)}{\|\phi \circ \bar{T}^{-1}\|_2(\|\phi \circ \bar{T}^{-1}\|_2 + 1)} \\
& = \frac{\phi \circ \bar{T}^{-1} - \phi}{\|\phi \circ \bar{T}^{-1}\|_2} - \frac{\phi(\|\phi \circ \bar{T}^{-1} - \phi\|_2^2 + 2\langle \phi \circ \bar{T}^{-1} - \phi, \phi \rangle)}{\|\phi \circ \bar{T}^{-1}\|_2(\|\phi \circ \bar{T}^{-1}\|_2 + 1)}.
\end{aligned}$$

Using the weak convergence of  $\sqrt{n}(\phi \circ \bar{T}^{-1} - \phi)$  to  $Y \times \phi'$  in the  $C[0, 1]$  topology, we have that

$$\sqrt{n}(\widehat{\phi} - \phi) \xrightarrow{d} Y \times \phi' - \frac{1}{2} \times 2\langle Y \times \phi', \phi \rangle \phi = Y \times \phi' - \langle Y \times \phi', \phi \rangle \phi$$

as  $n \rightarrow \infty$  in the  $C[0, 1]$  topology.

Finally, for the weak convergence of the  $\widehat{\xi}_i$ 's, observe that

$$\sqrt{n}(\widehat{\xi}_i - \xi_i) = \sqrt{n}\{\langle \widehat{X}_i - X_i, \widehat{\phi} - \phi \rangle + \langle \widehat{X}_i - X_i, \phi \rangle + \langle X_i, \widehat{\phi} - \phi \rangle\}$$

$$= \sqrt{n}\{\xi_i\langle(\phi \circ \bar{T}^{-1} - \phi), (\hat{\phi} - \phi)\rangle + \xi_i\langle(\phi \circ \bar{T}^{-1} - \phi), \phi\rangle + \xi_i\langle\phi, (\hat{\phi} - \phi)\rangle\}.$$

Using the independence of  $\xi_i$  and the  $T_j$ 's, and using the asymptotic distributions obtained above and in part (c), it follows that

$$\sqrt{n}(\hat{\xi}_i - \xi_i) \xrightarrow{d} \xi_i\{\langle Y \times \phi', \phi \rangle + \langle \phi, (Y \times \phi' - 2^{-1}\{\|Y \times \phi' + \phi\|_2^2 - 1\}\phi) \rangle\}$$

as  $n \rightarrow \infty$ .  $\square$

In order to prove Theorem 4, we will first prove a few crucial results.

**Proposition 1.** *Assume that  $\phi \in C^2[0, 1]$  and  $\inf_{t \in [0, 1]} T'(u) \geq \delta > 0$  almost surely for a deterministic constant  $\delta$ . Then, for each  $i \geq 1$ , we have  $\sum_{j=1}^{r-1} |\phi(s_{i,j+1}) - \phi(s_{i,j})| = \int_0^1 |\phi'(u)| du + B_{1,r}$  almost surely, where  $B_{1,r} = O(r^{-1})$  almost surely with the  $O(1)$  term being uniform in  $i$ . Further,  $\sum_{j \in \mathcal{J}_t} |\phi(s_{i,j+1}) - \phi(s_{i,j})| = \int_0^{T_i^{-1}(t)} |\phi'(u)| du + B_{2,r}(t)$  for all  $t \in [0, 1]$  almost surely, where  $\|B_{2,r}\|_\infty = O(r^{-1})$  almost surely with the  $O(1)$  term being uniform in  $i$ . Consequently, we have  $\sum_{j=1}^{r-1} |\phi(t_{j+1}) - \phi(t_j)| = \int_0^1 |\phi'(u)| du + B_{3,r}$  and  $\sum_{j \in \mathcal{J}_t} |\phi(t_{j+1}) - \phi(t_j)| = \int_0^t |\phi'(u)| du + B_{4,r}(t)$  for all  $t \in [0, 1]$  almost surely, where  $B_{3,r} = O(r^{-1})$  and  $\|B_{4,r}\|_\infty = O(r^{-1})$  almost surely.*

*Proof of Proposition 1.* First, let us define  $t_0 = 0$  and  $t_{r+1} = 1$  in case  $t_1 > 0$  and  $t_r < 1$ . Then,  $\{t_j : 0 \leq j \leq r+1\}$  is a partition of  $[0, 1]$ . Consider the sum  $S_i = \sum_{j=0}^r |\phi(s_{i,j+1}) - \phi(s_{i,j})|$  and note that by a Taylor expansion,  $S_i = \sum_{j=0}^r (s_{i,j+1} - s_{i,j}) |\phi'(\tilde{s}_{i,j})|$ , where  $\tilde{s}_{i,j} \in [s_{i,j}, s_{i,j+1}]$ . The right hand side is a Riemann sum approximation of  $\int_0^1 |\phi'(u)| du$  with  $\{s_{i,j} = T_i^{-1}(t_j) : 0 \leq j \leq r+1\}$  as the partition of  $[0, 1]$ , since  $T_i$  is a strictly increasing bijection. Thus, writing  $\Delta = \max_{0 \leq j \leq r} (s_{i,j+1} - s_{i,j})$ , we have

$$\begin{aligned} |S_i - \int_0^1 |\phi'(u)| du| &\leq \sup\{||\phi'(t)| - |\phi'(s)|| : s, t \in [0, 1] \text{ and } |t - s| \leq \Delta\} \\ &\leq \sup\{|\phi'(t) - \phi'(s)| : s, t \in [0, 1] \text{ and } |t - s| \leq \Delta\} \\ &\leq \|\phi''\|_\infty \Delta. \end{aligned}$$

Now for any  $0 \leq j \leq r$ , we have

$$s_{i,j+1} - s_{i,j} = T_i^{-1}(t_{j+1}) - T_i^{-1}(t_j) = (t_{j+1} - t_j)/T'_i(T_i^{-1}(\tilde{t}_j)),$$

for some  $\tilde{t}_j \in [t_j, t_{j+1}]$ . Using the assumption in the theorem and that on the grid, it now follows that  $\Delta = \max_{0 \leq j \leq r} (s_{i,j+1} - s_{i,j}) \leq \delta^{-1} O(r^{-1})$  uniformly on  $i$ . Thus,  $|S_i - \int_0^1 |\phi'(u)| du| \leq \|\phi''\|_\infty \delta^{-1} O(r^{-1})$ . To complete the first part of the proof, note that  $\sum_{j=1}^{r-1} |\phi(s_{i,j+1}) - \phi(s_{i,j})|$  differs from  $S_i$  by at most two terms, and both of these terms are  $O(r^{-1})$  uniformly over  $i$  by the same arguments as those for  $S_i$ .

For the second part, fix any  $t \in [0, 1]$ . Defining  $B_{2,r}(0) = 0$ , there is nothing to prove when  $t = 0$ . For  $t > 0$ , define  $t_0 = 0$ . If  $j^*$  is the largest  $j$  for which  $t_{j+1} \leq t$ , define  $t_{j^*+1} = t$  if  $t_{j^*+1} < t$ . Note that  $j^*$  depends on  $t$ . Then,  $\{t_j : 0 \leq j \leq j^*+1\}$  is a partition of  $[0, t]$ , and hence  $\{s_{i,j} = T_i^{-1}(t_j) : 0 \leq j \leq j^*+1\}$  is a partition of  $[0, T_i^{-1}(t)]$ . Define  $R_i(t) = \sum_{j=0}^{j^*} |\phi(s_{i,j+1}) - \phi(s_{i,j})|$ . Then, by similar arguments as earlier, we have

$$\left| R_i(t) - \int_0^{T_i^{-1}(t)} |\phi'(u)| du \right| \leq \|\phi''\|_\infty \delta^{-1} \max_{0 \leq j \leq j^*} (s_{i,j+1} - s_{i,j}) = B_{2,r}(t), \text{ say.}$$

Thus,  $\|B_{2,r}\|_\infty \leq O(r^{-1})$  uniformly over  $i$ . The proof is completed upon noting that  $R_i(t)$  differs from  $\sum_{j \in \mathcal{J}_t} |\phi(s_{i,j+1}) - \phi(s_{i,j})|$  by at most two terms, and both of them are  $O(r^{-1})$  uniformly over  $i$  by the same argument as before.

The last statement of the proposition is an immediate corollary for the case  $T = Id$  almost surely.  $\square$

Note that the  $B_{i,r}$ 's are not continuous functions, but we can still define their  $\|\cdot\|_\infty$  norms as all of them are uniformly bounded functions on  $[0, 1]$ . The following corollary is a consequence of Proposition 1 and the fact that  $\int_0^1 |\phi'(u)| du \in (0, \infty)$ .

**Corollary 1.** *Under the assumptions of Proposition 1, we have  $\tilde{F}_{i,d}(t) = \tilde{F}_i(t) + C_{1,r}(t)$  for all  $t \in [0, 1]$  almost surely for each  $i \geq 1$ , where  $\|C_{1,r}\|_\infty = O(r^{-1})$  almost surely uniformly over  $i$ . Further,  $F_d(t) = F_\phi(t) + C_{2,r}(t)$  for all  $t \in [0, 1]$ , where  $\|C_{2,r}\|_\infty = O(r^{-1})$ .*

**Lemma 2.** *Assume that  $\int_0^1 |\phi'(u)|^{-\epsilon} du < \infty$  for some  $\epsilon > 0$ . Then,  $|F_\phi^{-1}(s) - F_\phi^{-1}(t)| \leq C_\phi |t - s|^{\epsilon/(1+\epsilon)}$ , where  $C_\phi^{1+\epsilon} = \int_0^1 |\phi'(u)|^{-\epsilon} du$ . In other words,  $F_\phi^{-1}$  is  $\alpha$ -Hölder continuous for  $\alpha = \epsilon/(1 + \epsilon)$ .*

*Proof of Lemma 2.* Note that the assumption in the statement of the lemma implies that  $\phi' > 0$  almost everywhere with respect to the Lebesgue measure on  $[0, 1]$ . This fact along with Zarecki's theorem on the inverse of an absolutely continuous function (see, e.g., p. 271 in [Natanson \(1955\)](#)) applied to the function  $F_\phi$  yields that  $F_\phi^{-1}$  is absolutely continuous on  $[0, 1]$ . Thus,  $F_\phi^{-1}(t) = \int_0^t [F'_\phi(F_\phi^{-1}(u))]^{-1} du$ . Now, using Hölder's inequality and some algebraic manipulations, we obtain

$$|F_\phi^{-1}(s) - F_\phi^{-1}(t)| \leq \|\phi'\|_\infty |t - s|^{1/p} \left( \int_0^1 |\phi'(u)|^{-q+1} du \right)^{1/q}.$$

To complete the proof, choose  $q = 1 + \epsilon$ , which implies that  $p = (1 + \epsilon)/\epsilon$ .  $\square$

**Proposition 2.** *Assume that the conditions of Proposition 1 and Lemma 2 hold. Let  $\alpha = \epsilon/(1 + \epsilon)$  as in Lemma 2. Then, for each  $i \geq 1$ ,*

- (a)  $\tilde{F}_i^{-1}$  is  $\alpha$ -Hölder continuous almost surely.
- (b)  $\tilde{F}_{i,d}^{-1}(t) = \tilde{F}_i^{-1}(t) + \|T'_i\|_\infty D_{1,r}(t)$  for all  $t \in [0, 1]$  almost surely, where  $\|D_{1,r}\|_\infty = O(r^{-\alpha})$  almost surely uniformly over  $i$ .

*Proof of Proposition 2.* (a) Using the definition of  $\tilde{F}_i$ , it follows that

$$|\tilde{F}_i^{-1}(s) - \tilde{F}_i^{-1}(t)| = |T_i(F_\phi^{-1}(s)) - T_i(F_\phi^{-1}(t))| \leq \|T'_i\|_\infty |F_\phi^{-1}(s) - F_\phi^{-1}(t)| \leq \|T'_i\|_\infty C_\phi |s - t|^\alpha,$$

where the last inequality follows from Lemma 2. This completes the proof of part (a).

(b) As mentioned earlier,  $\tilde{F}_{i,d}$  is a cadlag step function with maximum jump discontinuities given by  $A_{i,r}$ . Thus, if  $t \in (\tilde{F}_{i,d}(t_j), \tilde{F}_{i,d}(t_{j+1})]$  for any  $1 \leq j \leq r-1$ , it follows that  $\tilde{F}_{i,d}(\tilde{F}_{i,d}^{-1}(t)) = \tilde{F}_{i,d}(t_{j+1}) = t + q_{i,j,r}(t)$ , where  $q_{i,j,r}(t) = \tilde{F}_{i,d}(t_{j+1}) - t$ . So,  $|q_{i,j,r}(t)| \leq \tilde{F}_{i,d}(t_{j+1}) - \tilde{F}_{i,d}(t_j) \leq A_{i,r}$ , where  $A_{i,r}$  is the maximum step size of  $\tilde{F}_{i,d}$  defined earlier. Now, from arguments similar to those used in Proposition 1, it follows that  $A_{i,r} = O(r^{-1})$  uniformly in  $i$ . Thus,  $\tilde{F}_{i,d}(\tilde{F}_{i,d}^{-1}(t)) = t + Q_{i,r}(t)$  for all  $t \in [0, 1]$  almost surely, where  $\|Q_r\|_\infty = O(r^{-1})$  almost surely uniformly over  $i$ .

From Proposition 1, we know that  $\tilde{F}_{i,d}(s) = \tilde{F}_i(s) + C_{1,r}(s)$  for all  $s \in [0, 1]$  almost surely, where  $\|C_{1,r}\|_\infty = O(r^{-1})$  almost surely uniformly over  $i$ . Letting  $s = \tilde{F}_{i,d}^{-1}(t)$ , we now have  $t + Q_r(t) = \tilde{F}_i(\tilde{F}_{i,d}^{-1}(t)) + C_{1,r}(\tilde{F}_{i,d}^{-1}(t))$  for all  $t$  almost surely. Re-arranging terms, we obtain  $\tilde{F}_{i,d}^{-1}(t) = \tilde{F}_i^{-1}(t + Q_{1,r}(t))$  for all  $t \in [0, 1]$  almost surely, where  $Q_{1,r}(t) = Q_r(t) - C_{1,r}(\tilde{F}_{i,d}^{-1}(t))$ . Thus,  $\|Q_{1,r}\|_\infty = O(r^{-1})$  almost surely uniformly over  $i$ . Now, using part (a), we can conclude that  $\tilde{F}_{i,d}^{-1}(t) = \tilde{F}_i^{-1}(t) + \|T'_i\|_\infty D_{1,r}(t)$  for all  $t \in [0, 1]$  almost surely, where  $D_{1,r}(t) = C_\phi |Q_{1,r}(t)|^\alpha$  satisfies  $\|D_{1,r}\|_\infty = O(r^{-\alpha})$  almost surely uniformly over  $i$ .  $\square$

*Proof of Theorem 4.* (a) Note that

$$\hat{F}_d^*(t) = n^{-1} \sum_{i=1}^n \tilde{F}_{i,d}^{-1}(t) = n^{-1} \sum_{i=1}^n \{ \tilde{F}_i^{-1}(t) + \|T'_i\|_\infty D_{1,r}(t) \} = \hat{F}_\phi^{-1}(t) + \left( n^{-1} \sum_{i=1}^n \|T'_i\|_\infty D_{1,r}(t) \right)$$

$$= \hat{F}_\phi^{-1}(t) + D_{2,r}(t)$$

for all  $t \in [0, 1]$  almost surely, where  $\|D_{2,r}\|_\infty = O(r^{-\alpha})$  almost surely since  $\|D_{1,r}\|_\infty = O(r^{-\alpha})$  almost surely and  $n^{-1} \sum_{i=1}^n \|T'_i\|_\infty = E(\|T'_1\|_\infty) + o(1)$  almost surely. Thus, it follows from Theorem 2.18 in Villani (2003) that

$$d_W^2(\hat{F}_d, F_\phi) = \|\hat{F}_d^* - F_\phi^{-1}\|_2^2 \leq 2\|\hat{F}_\phi^{-1} - F_\phi^{-1}\|_2^2 + 2\|D_{2,r}\|_2^2 \leq 2d_W^2(\hat{F}_\phi, F_\phi) + O(r^{-2\alpha})$$

almost surely. Combining the above statement with part (a) of Theorem 2 and 3 completes the proof of part (a) of Theorem 4.

(b) Next, note that

$$\begin{aligned} \hat{T}_{i,d}^*(t) &= n^{-1} \sum_{l=1}^n \tilde{F}_{l,d}^{-1}(\tilde{F}_{i,d}(t)) = n^{-1} \sum_{l=1}^n \left\{ \tilde{F}_l^{-1}(\tilde{F}_{i,d}(t)) + \|T'_i\|_\infty D_{1,r}(\tilde{F}_{i,d}(t)) \right\} \\ &= n^{-1} \sum_{l=1}^n \tilde{F}_l^{-1}(\tilde{F}_i(t) + C_{1,r}(t)) + n^{-1} \sum_{i=1}^n \|T'_i\|_\infty D_{1,r}(\tilde{F}_{i,d}(t)) \\ &= n^{-1} \sum_{l=1}^n \left[ \tilde{F}_l^{-1}(\tilde{F}_i(t)) + \left\{ \tilde{F}_l^{-1}(\tilde{F}_i(t) + C_{1,r}(t)) - \tilde{F}_l^{-1}(\tilde{F}_i(t)) \right\} \right] + n^{-1} \sum_{i=1}^n \|T'_i\|_\infty D_{1,r}(\tilde{F}_{i,d}(t)) \\ &= \hat{T}_i^{-1}(t) + n^{-1} \sum_{l=1}^n \left\{ \tilde{F}_l^{-1}(\tilde{F}_i(t) + C_{1,r}(t)) - \tilde{F}_l^{-1}(\tilde{F}_i(t)) \right\} + n^{-1} \sum_{i=1}^n \|T'_i\|_\infty D_{1,r}(\tilde{F}_{i,d}(t)), \end{aligned}$$

for all  $t \in [0, 1]$  almost surely. By part (a) of Proposition 2, we have  $|\tilde{F}_l^{-1}(\tilde{F}_i(t) + C_{1,r}(t)) - \tilde{F}_l^{-1}(\tilde{F}_i(t))| \leq \|T'_i\|_\infty D_{3,r}(t)$  for all  $t \in [0, 1]$  almost surely, where  $\|D_{3,r}\|_\infty = O(r^{-\alpha})$  almost surely uniformly over  $i$ . Thus,  $\sup_{t \in [0,1]} n^{-1} \sum_{i=1}^n |\tilde{F}_l^{-1}(\tilde{F}_i(t) + C_{1,r}(t)) - \tilde{F}_l^{-1}(\tilde{F}_i(t))| \leq \{E(\|T'_1\|_\infty) + o(1)\} O(r^{-\alpha})$  almost surely. Similar arguments yield  $\sup_{t \in [0,1]} n^{-1} \sum_{i=1}^n \|T'_i\|_\infty |D_{1,r}(\tilde{F}_{i,d}(t))| \leq \{E(\|T'_1\|_\infty) + o(1)\} O(r^{-\alpha})$  almost surely. Thus,

$$\hat{T}_{i,d}^*(t) = \hat{T}_i^{-1}(t) + D_{4,r}(t), \quad (7)$$

for all  $t \in [0, 1]$  almost surely, where  $\|D_{4,r}\|_\infty = O(r^{-\alpha})$  almost surely uniformly over  $i$ . Consequently,

$$\|\hat{T}_{i,d}^* - T_i^{-1}\|_\infty \leq \|\hat{T}_i^{-1} - T_i^{-1}\|_\infty + O(r^{-\alpha})$$

almost surely, where the  $O(1)$  term is uniform over  $i$ . This along with part (b) of Theorem 2 shows that  $\|\hat{T}_{i,d}^* - T_i^{-1}\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely for all  $i \geq 1$ . Equation (7) implies that  $\sqrt{n}(\hat{T}_{i,d}^* - T_i^{-1}) = \sqrt{n}(\hat{T}_i^{-1} - T_i^{-1}) + O(\sqrt{n}r^{-\alpha})$  in  $L_2[0, 1]$ . This in conjunction with part (b) of Theorem 3 proves that  $\sqrt{n}(\hat{T}_{i,d}^* - T_i^{-1})$  has the same asymptotic distribution as  $\sqrt{n}(\hat{T}_i^{-1} - T_i^{-1})$  in the  $L_2[0, 1]$  topology.

Next we consider  $\hat{T}_{i,d}(t) = \tilde{F}_{i,d}^{-1}(\hat{F}_d(t)) = \tilde{F}_i^{-1}(\hat{F}_d(t)) + \|T'_i\|_\infty D_{1,r}(\hat{F}_d(t))$  for all  $t \in [0, 1]$  almost surely (from part (b) of Proposition 2). Note that  $\hat{F}_d(t) = \{n^{-1} \sum_{l=1}^n \tilde{F}_{l,d}^{-1}\}^-(t) = \{G_n + D_{5,r}\}^-(t)$ , where  $G_n(s) = n^{-1} \sum_{l=1}^n \tilde{F}_l^{-1}(s)$  and  $D_{5,r}(s) = n^{-1} \sum_{l=1}^n \|T'_l\|_\infty D_{1,r}(s)$ . Thus,  $\|D_{5,r}\|_\infty = O(r^{-\alpha})$ . Also note that  $G_n$  is a strictly increasing homeomorphism on  $[0, 1]$ . Define  $\tilde{G}_{n,r} = G_n + D_{5,r} = n^{-1} \sum_{l=1}^n \tilde{F}_{l,d}^{-1}$  so that  $\tilde{G}_{n,r}$  is an increasing function (not necessarily strictly increasing) from  $[0, 1]$  onto  $[0, 1]$ . In fact, since each  $\tilde{F}_{l,d}^{-1}$  is left continuous and has right limits (being the generalized inverse of the cadlag function  $\tilde{F}_{l,d}$ ),  $\tilde{G}_{n,r}$  is also left continuous and has right limits.

If  $t \in (\tilde{G}_{n,r}(v), \tilde{G}_{n,r}(v+)]$  for some  $v \in [0, 1]$  with  $\tilde{G}_{n,r}(v+) > \tilde{G}_{n,r}(v)$ , then  $\tilde{G}_{n,r}(\hat{F}_d(t)) = \tilde{G}_{n,r}(\tilde{G}_{n,r}^{-1}(t)) = \tilde{G}_{n,r}(v) = t + (\tilde{G}_{n,r}(v) - t)$ . Now,  $|\tilde{G}_{n,r}(v) - t| \leq |\tilde{G}_{n,r}(v+) - \tilde{G}_{n,r}(v)| = |G_n(v+) - G_n(v) + D_{5,r}(v+) -$

$|D_{5,r}(v)| = |D_{5,r}(v+) - D_{5,r}(v)| = O(r^{-\alpha})$  uniformly in  $t$  almost surely, where the penultimate equality follows from the continuity of  $G_n$ . So, in these cases,  $G_n(\hat{F}_d(t)) = \tilde{G}_{n,r}(\hat{F}_d(t)) - D_{5,r}(\hat{F}_d(t)) = t + O(r^{-\alpha})$  uniformly in  $t$  almost surely, i.e.,  $t = G_n(\hat{F}_d(t)) + O(r^{-\alpha})$  uniformly in  $t$  almost surely.

Next, suppose that for some  $v_1 < v_2$ , we have  $\tilde{G}_{n,r}(v_1) = \tilde{G}_{n,r}(v_2)$ ,  $\tilde{G}_{n,r}(v) < \tilde{G}_{n,r}(v_1)$  for  $v < v_1$  and  $\tilde{G}_{n,r}(v) > \tilde{G}_{n,r}(v_2)$  for  $v > v_2$ . If  $t = \tilde{G}_{n,r}(v_1) = \tilde{G}_{n,r}(v_2)$ , then  $\tilde{G}_{n,r}(\hat{F}_d(t)) = t$  if  $v_1$  is a continuity point of  $\tilde{G}_{n,r}$ . If not, then this is already taken care of in the previous paragraph. In the former case, we have  $t = G_n(\hat{F}_d(t)) + O(r^{-\alpha})$  uniformly over  $t$  almost surely.

Finally, if  $t$  is a point of both continuity and strict increment of  $\tilde{G}_{n,r}$ , then  $\tilde{G}_{n,r}(\hat{F}_d(t)) = t$  as well, which implies that  $t = G_n(\hat{F}_d(t)) + O(r^{-\alpha})$  uniformly over  $t$  almost surely. Thus, all possibilities are exhausted. Let us denote the  $O(r^{-\alpha})$  term by  $D_{6,r}(\cdot)$ .

Now note that  $G_n^{-1} = (n^{-1} \sum_{l=1}^n \tilde{F}_l^{-1})^{-1} = (n^{-1} \sum_{l=1}^n T_l \circ F_\phi^{-1})^{-1} = F_\phi \circ \bar{T}^{-1}$ . Thus, it follows from our work above that  $\hat{F}_d(t) = F_\phi\{\bar{T}^{-1}(t - D_{6,r}(t))\}$ . Recall that  $\tilde{F}_i^{-1} = T_i \circ F_\phi^{-1}$  and that  $\hat{T}_{i,d}(t) = \tilde{F}_i^{-1}(\hat{F}_d(t)) + \|T'_i\|_\infty D_{1,r}(\hat{F}_d(t))$  for all  $t \in [0, 1]$  almost surely as obtained earlier. Since  $\hat{F}_d(t) = F_\phi\{\bar{T}^{-1}(t - D_{6,r}(t))\}$ , it follows from the decomposition of  $\hat{T}_{i,d}(t)$  that  $\hat{T}_{i,d}(t) = T_i\{\bar{T}^{-1}(t - D_{6,r}(t))\} + \|T'_i\|_\infty D_{1,r}(\hat{F}_d(t))$  for all  $t \in [0, 1]$  almost surely. Since  $\inf_{t \in [0,1]} T'(t) \geq \delta > 0$ , it follows that  $\inf_{t \in [0,1]} \bar{T}'(t) \geq n^{-1} \sum_{l=1}^n \inf_{t \in [0,1]} T'_l(t) \geq \delta > 0$ . So, by Taylor expansion, we have  $T_i\{\bar{T}^{-1}(t - D_{6,r}(t))\} = T_i(\bar{T}^{-1}(t)) + \|T'_i\|_\infty D_{7,r}(t)$  for all  $t \in [0, 1]$  almost surely, where  $\|D_{7,r}\|_\infty = O(r^{-\alpha})$  almost surely, where the  $O(1)$  term is uniform over  $i$ .

Combining the above findings, we arrive at

$$\begin{aligned} \hat{T}_{i,d}(t) &= \tilde{F}_i^{-1}(\hat{F}_d(t)) + \|T'_i\|_\infty D_{1,r}(\hat{F}_d(t)) = \tilde{F}_i^{-1}(G_n^{-1}(t) + D_{7,r}(t)) + \|T'_i\|_\infty D_{1,r}(\hat{F}_d(t)) \\ &= T_i(\bar{T}^{-1}(t)) + \|T'_i\|_\infty D_{7,r}(t) + \|T'_i\|_\infty D_{1,r}(\hat{F}_d(t)), \end{aligned}$$

where the last equality follows from the discussion in the previous paragraph. Since  $\|D_{1,r}\|_\infty = O(r^{-\alpha})$  almost surely uniformly over  $i$ , we obtain

$$\hat{T}_{i,d}(t) = \hat{T}_i(t) + \|T'_i\|_\infty D_{8,r}(t)$$

for all  $t \in [0, 1]$  almost surely, where  $\|D_{8,r}\|_\infty = O(r^{-\alpha})$  almost surely uniformly over  $i$ . Consequently,

$$\|\hat{T}_{i,d} - T_i\|_\infty \leq \|\hat{T}_i - T_i\|_\infty + O(1)r^{-\alpha},$$

almost surely. Combined with part (b) of Theorem 2, this shows that  $\|\hat{T}_{i,d} - T_i\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely for all  $i \geq 1$ . Equation (8) implies that  $\sqrt{n}(\hat{T}_{i,d} - T_i) = \sqrt{n}(\hat{T}_i - T_i) + O(\sqrt{nr}^{-\alpha})$  in  $L_2[0, 1]$ . This in conjunction with part (b) of Theorem 3 proves that  $\sqrt{n}(\hat{T}_{i,d} - T_i)$  has the same asymptotic distribution as  $\sqrt{n}(\hat{T}_i - T_i)$  in the  $L_2[0, 1]$  topology. This completes the proof of part (b) of Theorem 4.

(c) Next we register the warped functional observations. As mentioned earlier, since the warped observations are only recorded over a discrete grid, the registration algorithm in the fully observed case will not work. So, as a pre-processing step, we need to first smooth the warped discrete observations. We do this by using the Nadaraya-Watson kernel regression estimator as follows. Let  $k(\cdot)$  be any kernel supported on  $[-1, 1]$  and choose a bandwidth parameter  $h > 0$ . Then, the smooth version of  $\hat{X}_{i,d}$  is given by

$$X_i^\dagger(t) = \frac{\sum_{j=1}^r k\left(\frac{t-t_j}{h}\right) \tilde{X}_i(t_j)}{\sum_{j=1}^r k\left(\frac{t-t_j}{h}\right)} = \xi_i \frac{\sum_{j=1}^r k\left(\frac{t-t_j}{h}\right) \phi(T_i^{-1}(t_j))}{\sum_{j=1}^r k\left(\frac{t-t_j}{h}\right)}, \quad t \in [0, 1].$$

Now, note that

$$|X_i^\dagger(t) - \tilde{X}_i(t)| = \left| \xi_i \frac{\sum_{j=1}^r k\left(\frac{t-t_j}{h}\right) \{\phi(T_i^{-1}(t_j)) - \phi(T_i^{-1}(t))\}}{\sum_{j=1}^r k\left(\frac{t-t_j}{h}\right)} \right| \leq \|\phi'\|_\infty \delta^{-1} |\xi_i| \frac{\sum_{j=1}^r k\left(\frac{t-t_j}{h}\right) |t_j - t|}{\sum_{j=1}^r k\left(\frac{t-t_j}{h}\right)} \leq c |\xi_i| h,$$

for all  $t \in [0, 1]$  almost surely, where  $c$  is a constant not depending on  $i$  and  $t$ . The first inequality above follows from arguments similar to those used in the proof of Theorem 1. The second inequality follows from the fact that  $k(\cdot)$  is supported on  $[-1, 1]$  so that only those  $j$ 's in the numerator for which  $|t_j - t| \leq h$  will contribute to the sum. Thus,  $\|X_i^\dagger - \tilde{X}_i\|_\infty \leq c|\xi_i|h$  almost surely.

We register the warped discrete observation  $\tilde{X}_{i,d}$  by defining  $\hat{X}_i^* = X_i^\dagger \circ \hat{T}_{i,d}$  for each  $1 \leq i \leq n$ . Observe that

$$\begin{aligned} |\hat{X}_i^*(t) - \hat{X}_i(t)| &\leq |\hat{X}_i^*(t) - \tilde{X}_i(\hat{T}_{i,d}(t))| + |\tilde{X}_i(\hat{T}_{i,d}(t)) - \hat{X}_i(t)| \\ &\leq \|X_i^\dagger - \tilde{X}_i\|_\infty + |\xi_i| |\phi(T_i^{-1}(\hat{T}_{i,d}(t))) - \phi(T_i^{-1}(\hat{T}_i(t)))| \\ &\leq c|\xi_i|h + |\xi_i| |\phi(T_i^{-1}(\hat{T}_i(t) + \|T_i'\|_\infty D_{8,r}(t))) - \phi(T_i^{-1}(\hat{T}_i(t)))| \\ &\leq c|\xi_i|h + |\xi_i| \|T_i'\|_\infty |D_{8,r}(t)| \|\phi'\|_\infty \delta^{-1} \leq O(1)|\xi_i|(h + \|T_i'\|_\infty r^{-\alpha}) \end{aligned} \quad (8)$$

for all  $t \in [0, 1]$  almost surely, where the  $O(1)$  term is uniform in  $i$  and  $t$ . The last two inequalities above follow from a first order Taylor expansion and the fact that  $\|D_{8,r}\|_\infty = O(r^{-\alpha})$  almost surely uniformly over  $i$ . Hence,

$$\|\hat{X}_i^* - \hat{X}_i\|_\infty = O(1)|\xi_i|(h + \|T_i'\|_\infty r^{-\alpha})$$

almost surely. In conjunction with part (c) of Theorem 2, this shows that  $\|\hat{X}_i^* - X_i\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely for all  $i \geq 1$ . Equation (8) implies that  $\sqrt{n}(\hat{X}_i^* - X_i) = \sqrt{n}(\hat{X}_i - X_i) + O(\sqrt{n}(h + r^{-\alpha}))$  in  $L_2[0, 1]$ . Invoking part (c) of Theorem 3 thus establishes that  $\sqrt{n}(\hat{X}_i^* - X_i)$  has the same asymptotic distribution as  $\sqrt{n}(\hat{X}_i - X_i)$  in the  $L_2[0, 1]$  topology. This completes the proof of part (c) of Theorem 4. (d) Next, define the random measure induced by  $\hat{X}_i^*$  as

$$\begin{aligned} \hat{F}_i^*(t) &= \sum_{j \in \mathcal{J}_t} |\hat{X}_i^*(t_{j+1}) - \hat{X}_i^*(t_j)| \bigg/ \sum_{j=1}^{r-1} |\hat{X}_i^*(t_{j+1}) - \hat{X}_i^*(t_j)| \\ &= \sum_{j \in \mathcal{J}_t} |\hat{X}_i^\dagger(\hat{T}_{i,d}(t_{j+1})) - \hat{X}_i^\dagger(\hat{T}_{i,d}(t_j))| \bigg/ \sum_{j=1}^{r-1} |\hat{X}_i^\dagger(\hat{T}_{i,d}(t_{j+1})) - \hat{X}_i^\dagger(\hat{T}_{i,d}(t_j))| \\ &= \left\{ \sum_{j \in \mathcal{J}_t} |\tilde{X}_i(\hat{T}_{i,d}(t_{j+1})) - \tilde{X}_i(\hat{T}_{i,d}(t_j))| + O(h)|\xi_i| \right\} \bigg/ \left\{ \sum_{j=1}^{r-1} |\tilde{X}_i(\hat{T}_{i,d}(t_{j+1})) - \tilde{X}_i(\hat{T}_{i,d}(t_j))| + O(h)|\xi_i| \right\} \end{aligned}$$

for all  $t \in [0, 1]$  almost surely, where the  $O(1)$  term is uniform in  $i$  and  $t$ , and the last equality follows from the fact that  $\|X_i^\dagger - \tilde{X}_i\|_\infty \leq c|\xi_i|h$  almost surely. Also note that by definition of  $\tilde{X}_i$ , the term  $|\xi_i|$  cancels from the numerator and the denominator.

Using the fact that  $\hat{T}_{i,d}(t) = \hat{T}_i(t) + \|T_i'\|_\infty D_{8,r}(t)$  with  $\|D_{8,r}\|_\infty = O(r^{-\alpha})$  almost surely, and arguments similar to those used in the proof of Proposition 1, one obtains

$$\hat{F}_i^*(t) = \hat{F}_\phi(t) + O(1)(h + \|T_i'\|_\infty r^{-\alpha})$$

for all  $t \in [0, 1]$  almost surely, where the  $O(1)$  term is uniform in  $i$  and  $t$  almost surely. Now, using Lemma 2 and arguments similar to those used in the proof of part (b) of Proposition 2, we have

$$(\hat{F}_i^*)^-(t) = \hat{F}_\phi^{-1}(t) + O(1)r^{-\alpha}(h + \|T_i'\|_\infty r^{-\alpha})$$

for all  $t \in [0, 1]$  almost surely, where the  $O(1)$  term is uniform in  $i$  and  $t$  almost surely. Thus,

$$d_W^2(\hat{F}_i^*, F_\phi) = \|(\hat{F}_i^*)^- - F_\phi^{-1}\|_2^2 \leq 2\|\hat{F}_\phi^{-1} - F_\phi^{-1}\|_2^2 + O(1)r^{-2\alpha}(h^2 + r^{-2\alpha})$$



$$= 2d_W^2(\hat{F}_\phi, F_\phi) + O(1)r^{-2\alpha}(h^2 + r^{-2\alpha})$$

almost surely. Combining the above statement with part (d) of Theorems 2 and 3 completes the proof of part (d) of Theorem 4.

(e) Next, define  $\bar{X}_{r*} = n^{-1} \sum_{i=1}^n \hat{X}_i^*$ . Since  $\|\hat{X}_i^* - \hat{X}_i\|_\infty = O(1)|\xi_i|(h + \|T'_i\|_\infty r^{-\alpha})$  almost surely, it follows that

$$\begin{aligned} \|(\bar{X}_{r*} - \mu) - (\bar{X}_r - \mu)\|_\infty &\leq n^{-1} \sum_{i=1}^n \|\hat{X}_i^* - \hat{X}_i\|_\infty \leq O(1)\{h + r^{-\alpha} n^{-1} \sum_{i=1}^n \|T'_i\|_\infty\} \\ &\leq O(1)(h + r^{-\alpha}) \end{aligned} \quad (9)$$

almost surely since  $E(\|T'_1\|_\infty) < \infty$ . Along with part (e) of Theorem 2, this shows that  $\|\bar{X}_{r*} - \mu\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Equation (9) implies that  $\sqrt{n}(\bar{X}_{r*} - \mu) = \sqrt{n}(\bar{X}_r - \mu) + O(\sqrt{n}(h + r^{-\alpha}))$  in  $L_2[0, 1]$ . So by part (e) of Theorem 3 we see that  $\sqrt{n}(\bar{X}_{r*} - \mu)$  has the same asymptotic distribution as  $\sqrt{n}(\bar{X}_r - \mu)$  in the  $L_2[0, 1]$  topology, and the proof of part (e) of Theorem 4 is complete.

(f) Next, we consider the empirical covariance operator of the  $\hat{X}_i^*$ 's which we will denote by  $\widehat{\mathcal{K}}_{r*} = n^{-1} \sum_{i=1}^n (\hat{X}_i^* - \bar{X}_{r*}) \otimes (\hat{X}_i^* - \bar{X}_{r*})$ . Recall  $S_1 = n^{-1} \sum_{i=1}^n (\hat{X}_i - \mu) \otimes (\hat{X}_i - \mu)$  from the proof of part (f) of Theorem 3. Now, some straightforward manipulations yield

$$\begin{aligned} \widehat{\mathcal{K}}_{r*} &= S_1 + n^{-1} \sum_{i=1}^n (\hat{X}_i^* - \hat{X}_i) \otimes (\hat{X}_i^* - \hat{X}_i) - (\bar{X}_{r*} - \mu) \otimes (\bar{X}_{r*} - \mu) \\ &\quad + n^{-1} \sum_{i=1}^n \{(\hat{X}_i^* - \hat{X}_i) \otimes (\hat{X}_i - \mu) + (\hat{X}_i - \mu) \otimes (\hat{X}_i^* - \hat{X}_i)\} \\ &= S_1 + W_1 - W_2 + W_3, \quad \text{say.} \end{aligned}$$

Note that  $\|W_1\| \leq n^{-1} \sum_{i=1}^n \|\hat{X}_i^* - \hat{X}_i\|_2^2 \leq O(1)\{h^2 n^{-1} \sum_{i=1}^n |\xi_i|^2 + r^{-2\alpha} n^{-1} \sum_{i=1}^n \|T'_i\|_\infty^2\} = O(1)(h^2 + r^{-2\alpha})$  almost surely. Next, from the previous paragraph, it follows that  $\|W_2\| \leq \|\bar{X}_{r*} - \mu\|_2^2 \leq O(1)(h^2 + r^{-2\alpha}) + 2\|\bar{X}_r - \mu\|_\infty^2$ . Moreover,  $\|W_3\| \leq 2n^{-1} \sum_{i=1}^n \|\hat{X}_i^* - \hat{X}_i\|_2 \|\hat{X}_i - \mu\|_2 \leq O(1)n^{-1} \sum_{i=1}^n \{h|\xi_i| + \|T'_i\|_\infty r^{-\alpha}\} \|\hat{X}_i - \mu\|_2$  almost surely. Observe that

$$\begin{aligned} n^{-1} \sum_{i=1}^n |\xi_i| \|\hat{X}_i - \mu\|_2 &= n^{-1} \sum_{i=1}^n |\xi_i| \|\xi_i \phi \circ \bar{T}^{-1} - E(\xi_1) \phi\|_2 \\ &\leq n^{-1} \sum_{i=1}^n |\xi_i| |\xi_i - E(\xi_1)| \|\phi \circ \bar{T}^{-1}\|_2 + n^{-1} \sum_{i=1}^n |\xi_i| |E(\xi_1)| \|\phi \circ \bar{T}^{-1} - \phi\|_2. \end{aligned}$$

Since  $\|\phi \circ \bar{T}^{-1} - \phi\|_\infty \rightarrow 0$  almost surely, it follows that the first term above is  $O(1)$  almost surely, and the second term is  $o(1)$  almost surely. Similar arguments show that  $n^{-1} \sum_{i=1}^n \|T'_i\|_\infty \|\hat{X}_i - \mu\|_2 = O(1)$  almost surely. Thus,  $\|W_3\| \leq O(1)(h + r^{-\alpha})$  almost surely. Also,  $S_2$  in the proof of part (f) of Theorem 3 satisfies  $\|S_2\| = O_P(n^{-1})$ . Combining the above facts and using the decomposition of  $\widehat{\mathcal{K}}_r$  in the proof of part (f) of Theorem 3, it follows that

$$\widehat{\mathcal{K}}_{r*} = S_1 + O(1)(h + r^{-\alpha} + \|\bar{X}_r - \mu\|_\infty^2) = \widehat{\mathcal{K}}_r + O(1)(h + r^{-\alpha} + \|\bar{X}_r - \mu\|_\infty^2) \quad (10)$$

almost surely. This along with part (f) of Theorem 2 shows that  $\|\widehat{\mathcal{K}}_{r*} - \mathcal{K}\| \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. By part (e) of Theorem 3, it follows that  $\sqrt{n}\|\bar{X}_r - \mu\|_\infty = O_P(1)$  as  $n \rightarrow \infty$ . So, equation (10) implies that  $\sqrt{n}(\widehat{\mathcal{K}}_{r*} - \mathcal{K}) = \sqrt{n}(\widehat{\mathcal{K}}_r - \mathcal{K}) + O(\sqrt{n}(h + r^{-\alpha}))$  in  $L_2[0, 1]$ . This in conjunction with part (f) of Theorem 3 proves that  $\sqrt{n}(\widehat{\mathcal{K}}_{r*} - \mathcal{K})$  has the same asymptotic distribution as  $\sqrt{n}(\widehat{\mathcal{K}}_r - \mathcal{K})$  in

the Hilbert-Schmidt topology.

For the convergence of the empirical covariance kernel  $\hat{K}_{r*}(s, t) = n^{-1} \sum_{i=1}^n [\hat{X}_i^*(s) - \bar{X}_{r*}(s)][\hat{X}_i^*(t) - \bar{X}_{r*}(t)]$ , we follow the same decomposition as above for the case of the operator. Noting the all the bounds used for that proof remain valid in the sup-norm and using the same arguments, we arrive that

$$\hat{K}_{r*}(s, t) = \hat{K}_r(s, t) + O(1)(h + r^{-\alpha} + \|\bar{X}_r - \mu\|_\infty^2) \quad (11)$$

for all  $s, t \in [0, 1]$  almost surely, where the  $O(1)$  term is uniform in  $s, t$  almost surely. This along with part (f) of Theorem 2 shows that  $\|\hat{K}_{r*} - K\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Equation (11) implies that  $\{\sqrt{n}(\hat{K}_{r*}(s, t) - K(s, t)) : s, t \in [0, 1]\} = \{\sqrt{n}(\hat{K}_r(s, t) - K(s, t)) : s, t \in [0, 1]\} + O(\sqrt{n}(h + r^{-\alpha}))$  in  $L_2[0, 1]$  with the  $O(1)$  term being uniform in  $s, t$ . This in conjunction with part (f) of Theorem 3 proves that  $\{\sqrt{n}(\hat{K}_{r*}(s, t) - K(s, t)) : s, t \in [0, 1]\}$  has the same asymptotic distribution as  $\{\sqrt{n}(\hat{K}_r(s, t) - K(s, t)) : s, t \in [0, 1]\}$  in the  $L_2([0, 1]^2)$  topology.

To prove the strong consistency and the weak convergence of the estimated eigenfunction, we will use perturbation bounds for compact operators (see, e.g., Ch. 5 of Hsing and Eubank (2015)). The leading eigenfunction  $\hat{\phi}_*$  of  $\hat{\mathcal{K}}_{r*}$  satisfies the inequality  $\|\hat{\phi}_* - \phi\|_2 \leq 2\sqrt{2}\lambda^{-1}\|\hat{\mathcal{K}}_{r*} - \mathcal{K}\| \rightarrow 0$  as  $n \rightarrow \infty$  almost surely. Further, Theorem 5.1.8 of Hsing and Eubank (2015), specifically equation (5.27), implies that  $\sqrt{n}(\hat{\phi}_* - \phi)$  has the same asymptotic distribution (in  $L_2[0, 1]$ ) as that of  $\mathcal{S}\sqrt{n}(\hat{\mathcal{K}}_{r*} - \mathcal{K})\phi$ , where, in our setup,  $\mathcal{S} = -\lambda^{-1}(\mathcal{I} - \phi \otimes \phi)$  with  $\lambda = \text{Var}(\xi_1)$  being the leading eigenvalue of  $\mathcal{K}$ , and  $\mathcal{I}$  being the identity operator. Thus, from the results already establishes, it follows that the asymptotic distribution of  $\sqrt{n}(\hat{\phi}_* - \phi)$  is that of  $-\lambda^{-1}(\mathcal{I} - \phi \otimes \phi)\sqrt{n}(\hat{\mathcal{K}}_r - \mathcal{K})\phi$ . Using the expression of the asymptotic distribution of  $\sqrt{n}(\hat{\mathcal{K}}_r - \mathcal{K})$  obtained in part (f) of Theorem 3 and some simple calculations, it follows that the asymptotic distribution of  $\sqrt{n}(\hat{\phi}_* - \phi)$  is that of  $Y \times \phi' - \langle Y \times \phi', \phi \rangle \phi$ , which is the same as in Theorem 3.

The proof of the strong consistency and the weak convergence of  $\hat{\xi}_{i*}$  follows in direct analogy to that of  $\hat{\xi}_i$  upon using part (c) and the above facts. The proof of part (f) of Theorem 4 is now complete.  $\square$

*Proof of Theorem 5.* First observe that

$$\begin{aligned} |\tilde{F}_{i,w}(t) - \tilde{F}_i(t)| &\leq \left| \frac{\int_0^t |\hat{X}_{i,w}^{(1)}(u)| du}{\int_0^1 |\hat{X}_{i,w}^{(1)}(u)| du} - \frac{\int_0^t |\hat{X}_{i,w}^{(1)}(u)| du}{\int_0^1 |\hat{X}_i^{(1)}(u)| du} \right| + \left| \frac{\int_0^t |\hat{X}_{i,w}^{(1)}(u)| du}{\int_0^1 |\hat{X}_{i,w}^{(1)}(u)| du} - \frac{\int_0^t |\tilde{X}_i'(u)| du}{\int_0^1 |\tilde{X}_i'(u)| du} \right| \\ &\leq \frac{2 \int_0^1 |\hat{X}_{i,w}^{(1)}(u) - \tilde{X}_i'(u)| du}{\int_0^1 |\tilde{X}_i'(u)| du} \leq \frac{2 \|\hat{X}_{i,w}^{(1)} - \tilde{X}_i'\|_2}{|\xi_i| \int_0^1 |\phi'(u)| du} = d_\phi |\xi_i|^{-1} A_{i,r}, \quad \text{say.} \\ \Rightarrow \|\tilde{F}_{i,w} - \tilde{F}_i\|_\infty &\leq d_\phi |\xi_i|^{-1} A_{i,r}. \end{aligned} \quad (12)$$

Since the term  $A_{i,r}$  will be key for our proof, we will first bound  $E\{A_{i,r}^2\}$ . To achieve this, we will first provide bounds on  $E\{A_{i,r}^2|\xi_i, T_i\}$  using standard tools from non-parametric regression. So, we will have to estimate the MSE for the regression problem  $Y_{ij} = \xi_i \phi(T_i^{-1}(t_j)) + \epsilon_{ij}$  and integrate this MSE over  $u \in [0, 1]$ , when  $\xi_i$  and  $T_i$  are fixed. The expression for the MSE in the deterministic design case is the same as the conditional MSE (given design points) in the random design case with the design distribution being uniform on  $[0, 1]$ . Next, observe that  $\text{Var}(\hat{X}_{i,w}(u)|\xi_i, T_i)$  does not depend on  $\xi_i$  and  $T_i$  and is thus uniform over  $i$  (since the  $\epsilon_{ij}$ 's are i.i.d.). For  $u \in [h_1, 1 - h_1]$ , the expression of this variance is given in p. 137 in Wand and Jones (1995) and equals  $O((rh_1)^{-3})$ , where the  $O(1)$  term depends on  $k_1$ , is bounded and is uniform over  $u \in [h_1, 1 - h_1]$ . Next, we have to take into account the boundary points. Let  $u = \alpha h_1$  for some  $\alpha \in [0, 1)$ . It follows from a similar analysis that even in this case,

$\text{Var}(\hat{X}_{i,w}(u)|\xi_i, T_i) = O((rh_1)^{-3})$ , where the  $O(1)$  term is integrable over  $\alpha \in [0, 1]$  (see, e.g. pp. 244-247 in Schimek (2000)). Similar estimates also hold for  $t \in [1 - h_1, 1]$ , say  $t = 1 - \alpha h_1$ . Hence, we get that  $\text{Var}(\hat{X}_{i,w}(u)|\xi_i, T_i) = O((rh_1)^{-3})$  for all  $u \in [0, 1]$  with the  $O(1)$  term being integrable over  $u \in [0, 1]$ .

Next we consider the bias. In our case the degree of the fitted polynomial is one more than the degree of derivative estimated. Thus, applying Taylor's formula and using the expressions in Thm. 9.1 and pp. 244-247 in Schimek (2000), we have  $|\text{Bias}(\hat{X}_{i,w}(u)|\xi_i, T_i)| = \|\tilde{X}_i^{(3)}\|_\infty O(h_1^2) + \|\tilde{X}_i^{(4)}\|_\infty o(h_1^2)$  for all  $u \in [0, 1]$ . Here, the  $O(1)$  and  $o(1)$  terms are non-random and are integrable in  $u \in [0, 1]$ . So, using the moment assumptions on the sup-norm of the derivatives of  $T$ , the independence of the  $\xi_i$ 's and the  $T_i$ 's along with the assumption that  $\inf_{t \in [0,1]} T'(t) \geq \delta > 0$ , it follows that

$$E\{A_{i,r}^2\} = O(h_1^4) + O((rh_1^3)^{-1}) \quad (13)$$

where the  $O(1)$  terms are bounded and do not depend on  $i$  (the  $\tilde{X}_i$ 's are i.i.d). This also implies (using Markov's inequality) that

$$n^{-1} \sum_{i=1}^n A_{i,r}^2 = O_P(h_1^4 + (rh_1^3)^{-1}) \quad (14)$$

We will now proceed with the rest of the proof. First, let  $u_{i,t} = \tilde{F}_{i,w}^{-1}(t)$ . From (12), it follows that  $\tilde{F}_i(u_{i,t}) = t - \tilde{A}_{i,r}(t)$ , where  $\|\tilde{A}_{i,r}\|_\infty \leq d_\phi |\xi_i|^{-1} A_{i,r}$ . Thus, using part (a) of Proposition 2, it follows that  $|\tilde{F}_{i,w}^{-1}(t) - \tilde{F}_i^{-1}(t)| = |u_{i,t} - \tilde{F}_i^{-1}(t)| = |\tilde{F}_i^{-1}(t - \tilde{A}_{i,r}(t)) - \tilde{F}_i^{-1}(t)| \leq \|T'_i\|_\infty c'_\phi |\xi_i|^{-\alpha} A_{i,r}^\alpha$  for a constant  $c'_\phi$ . So,  $\|\tilde{F}_{i,w}^{-1} - \tilde{F}_i^{-1}\|_\infty \leq \|T'_i\|_\infty c'_\phi |\xi_i|^{-\alpha} A_{i,r}^\alpha$ . Thus,  $\hat{F}_{\phi,e}^{-1} = n^{-1} \sum_{i=1}^n \tilde{F}_{i,w}^{-1} = n^{-1} \sum_{i=1}^n \tilde{F}_i^{-1} + \tilde{B}_r = \hat{F}_\phi^{-1} + \tilde{B}_r$ , where  $\|\tilde{B}_r\|_\infty \leq c'_\phi n^{-1} \sum_{i=1}^n \|T'_i\|_\infty |\xi_i|^{-\alpha} A_{i,r}^\alpha$ . Define  $R_r = n^{-1} \sum_{i=1}^n \|T'_i\|_\infty |\xi_i|^{-\alpha} A_{i,r}^\alpha$ . By Hölder's inequality, the law of large numbers, independence of  $T_i$ 's and  $\xi_i$ 's, and (14), we get that

$$\begin{aligned} R_r &\leq \left[ n^{-1} \sum_{i=1}^n \|T'_i\|_\infty^{2/(2-\alpha)} |\xi_i|^{-2\alpha/(2-\alpha)} \right]^{1-\alpha/2} \left[ n^{-1} \sum_{i=1}^n A_{i,r}^2 \right]^{\alpha/2} \\ \Rightarrow R_r &= O_P(h_1^{2\alpha} + (rh_1^3)^{-\alpha/2}) \end{aligned} \quad (15)$$

(a) Since  $d_W^2(\hat{F}_{\phi,e}, F_\phi) = \|\hat{F}_{\phi,e}^{-1} - F_\phi^{-1}\|_2^2 \leq 2\|\hat{F}_{\phi,e}^{-1} - \hat{F}_\phi^{-1}\|_2^2 + 2\|\hat{F}_\phi^{-1} - F_\phi^{-1}\|_2^2 \leq 2R_r^2 + 2d_W^2(\hat{F}_\phi, F_\phi)$ , the proof follows using part (a) of Theorem 3 and (15).

(b) Note that  $\hat{T}_{i,e}^{-1}(t) = \hat{F}_{\phi,e}^{-1}(\tilde{F}_{i,w}(t)) = \hat{F}_\phi^{-1}(\tilde{F}_{i,w}(t)) + \tilde{B}_r(\tilde{F}_{i,w}(t))$  using statements proved earlier. Now, arguments in the proof of part (b) of Theorem 3 along with (12) yield  $\hat{F}_\phi^{-1}(\tilde{F}_{i,w}(t)) = \hat{T}_i^{-1}(t) + \tilde{C}_r(t)$ , where  $\|\tilde{C}_r\|_\infty \leq \text{const.} R_r$ . Thus,  $\hat{T}_{i,e}^{-1} = \hat{T}_i^{-1} + \tilde{C}_{1,r}$ , where  $\|\tilde{C}_{1,r}\|_\infty \leq \text{const.} R_r$ . The proof of the first statement in part (b) of this theorem now follows using part (b) of Theorem 3 and (15).

Next consider  $\hat{T}_{i,e}(t) = \tilde{F}_{i,w}^{-1}(\hat{F}_{\phi,e}(t)) = \tilde{F}_i^{-1}(\hat{F}_{\phi,e}(t)) + \tilde{C}_{2,r,i}(t)$ , where  $\|\tilde{C}_{2,r,i}\|_\infty \leq \|T'_i\|_\infty c'_\phi |\xi_i|^{-\alpha} A_{i,r}^\alpha$  from statements proved earlier. Note that if  $\hat{F}_{\phi,e}(t) = v$  then  $t = \hat{F}_{\phi,e}^{-1}(v) = \hat{F}_\phi^{-1}(v) + \tilde{C}_{3,r}(v)$ , where  $\|\tilde{C}_{3,r}\|_\infty \leq R_r$ . So,  $\hat{F}_{\phi,e}(t) = v = \hat{F}_\phi(t - \tilde{C}_{3,r}(v)) = F_\phi(\bar{T}^{-1}(t - \tilde{C}_{3,r}(v)))$ . Noting that  $\tilde{F}_i^{-1} = T_i \circ F_\phi^{-1}$ , we get that  $\tilde{F}_i^{-1}(\hat{F}_{\phi,e}(t)) = T_i(\bar{T}^{-1}(t - \tilde{C}_{3,r}(v))) = T_i(\bar{T}^{-1}(t)) + \|T'_i\|_\infty \tilde{C}_{4,r}(v) = \tilde{F}_i^{-1}(\hat{F}_\phi(t)) + \|T'_i\|_\infty \tilde{C}_{4,r}(v) = \hat{T}_i(t) + \|T'_i\|_\infty \tilde{C}_{4,r}(v)$ , where  $\|\tilde{C}_{4,r}\|_\infty \leq R_r$ . This follows from arguments similar to those used earlier using the smoothness of  $T$  and the assumption that  $\inf_{t \in [0,1]} T'(t) \geq \delta > 0$ . Thus, we finally have

$$\|\hat{T}_{i,e} - \hat{T}_i\|_\infty \leq \text{const.} \{ \|T'_i\|_\infty R_r + \|T'_i\|_\infty |\xi_i|^{-\alpha} A_{i,r}^\alpha \}. \quad (16)$$

The proof of the second statement of part (b) of this theorem is now completed via part (b) of Theorem 3, (13) and (15).

For proving part (c) of the theorem we will first have to control  $E\{\|\hat{X}_{i,w} - \tilde{X}_i\|_\infty^2 | \xi_i, T_i\}$  for each  $i$ . Recall that

$$\hat{X}_{i,w}(t) = \frac{1}{r} \sum_{j=1}^r \frac{\{\hat{s}_2(t; h_2) - \hat{s}_1(t; h_2)(t_j - t)\} k_{2,h_2}(t_j - t) Y_{ij}}{\hat{s}_2(t; h_2) \hat{s}_0(t; h_2) - \hat{s}_1^2(t; h_2)},$$

where  $k_{2,h_2}(u) = h_2^{-1} k_2(u/h_2)$  and  $\hat{s}_l(t; h_2) = r^{-1} \sum_{j=1}^r (t_j - t)^l k_{2,h_2}(t_j - t)$  for  $l = 0, 1, 2$ . Call the denominator  $\hat{f}(t)$ , which is deterministic. We will first analyse the term  $\tilde{Y}_{i,w}(t)$  which is defined like  $\hat{X}_{i,w}(t)$  but with  $\tilde{X}_i(t_j)$  in place of  $Y_{ij}$ . Define  $\tilde{Z}_{i,w}(t) = \hat{X}_{i,w}(t) - \tilde{Y}_{i,w}(t)$ .

Using Taylor's formula, we get that  $\tilde{X}_i(t_j) = \tilde{X}_i(t) + (t_j - t) \tilde{X}_i'(t) + 2^{-1}(t_j - t)^2 \tilde{X}_i''(t) + 6^{-1}(t_j - t)^3 \tilde{X}_i^{(3)}(\tilde{t}_{i,j})$ , where  $\tilde{t}_{i,j}$  lies between  $t$  and  $t_j$ . Plugging-in this expansion in the definition of  $\tilde{Y}_{i,w}(t)$ , we have

$$\begin{aligned} \tilde{Y}_{i,w}(t) &= \tilde{X}_i(t) + \frac{\tilde{X}_i''(t) \hat{s}_2^2(t; h_2) - \hat{s}_1(t; h_2) \hat{s}_3(t; h_2)}{2 \hat{s}_2(t; h_2) \hat{s}_0(t; h_2) - \hat{s}_1^2(t; h_2)} \\ &\quad + \frac{1}{6r} \sum_{j=1}^r \frac{\{\hat{s}_2(t; h_2) - \hat{s}_1(t; h_2)(t_j - t)\} k_{2,h_2}(t_j - t) (t_j - t)^3 \tilde{X}_i^{(3)}(\tilde{t}_{i,j})}{\hat{s}_2(t; h_2) \hat{s}_0(t; h_2) - \hat{s}_1^2(t; h_2)} \\ &= \tilde{X}_i(t) + Q_{i,1}(t; h_2) + Q_{i,2}(t; h_2), \quad \text{say} \end{aligned}$$

for all  $t \in [0, 1]$ . Note that the term involving  $\tilde{X}_i'(t)$  vanishes, which plays a crucial role in putting the local linear estimator at an advantage over other standard non-parametric regression estimators near the boundary of the data set. Thus,  $|\hat{X}_{i,w}(t) - \tilde{X}_i(t)| \leq |\tilde{Y}_{i,w}(t) - \tilde{X}_i(t)| + |\tilde{Z}_{i,w}(t)| \leq |Q_{i,1}(t; h_2)| + |Q_{i,2}(t; h_2)| + |\tilde{Z}_{i,w}(t)|$ .

By approximations of Riemann sums, we have  $\hat{s}_l(t; h_2) = h_2^l \int_{-1}^1 u^l k_2(u) du + O((rh_2)^{-1})$  uniformly for  $t \in [h_2, 1 - h_2]$ . Also, for  $t \in [0, h_2]$ , say,  $t = \alpha h_2$  with  $\alpha \in [0, 1]$ , we have  $\hat{s}_l(t; h_2) = h_2^l \int_{-\alpha}^1 u^l k_2(u) du + O((rh_2)^{-1})$  uniformly for  $\alpha \in [0, 1]$ . The same estimate also holds for  $t \in (1 - h_2, 1]$ , say,  $t = 1 - \alpha h_2$ . Define  $\mu_{l,\alpha} = \int_{-\alpha}^1 u^l k_2(u) du$  for  $l = 0, 1, 2$ . These estimates imply that for  $t \in [h_2, 1 - h_2]$ , we have  $|Q_{i,2}(t; h_2)| \leq 2^{-1} \|\tilde{X}_i''\|_\infty \{h_2^2 \int_{-1}^1 u^2 k_2(u) du + O((rh_2)^{-1})\}$ . Further, for boundary points, we have  $|Q_{i,2}(t; h_2)| \leq 2^{-1} \|\tilde{X}_i''\|_\infty \{h_2^2 |B_\alpha| + O((rh_2)^{-1})\}$  for  $\alpha \in [0, 1]$ , where  $B_\alpha = [\mu_{2,\alpha}^2 - \mu_{1,\alpha} \mu_{3,\alpha}] / [\mu_{2,\alpha} \mu_{0,\alpha} - \mu_{1,\alpha}^2]$ . In both case, the  $O(1)$  terms are non-random (hence does not depend on  $i$ ) and uniform over choices of  $t$ . Note that the leading term in the squared bias term obtainable from the previous bias expression is an upper bound for the coefficient of the squared bias term in the general result obtained in Thm. 3.3 in [Fan and Gijbels \(1996\)](#). It can be shown using similar arguments that  $|Q_{i,3}(t; h_2)| \leq \|\tilde{X}_i^{(3)}\|_\infty o(h_2^2)$ , where the  $o(1)$  term is non-random and uniform over  $t \in [0, 1]$ . Note that for  $\alpha = 1$ , which correspond to  $t \in [h_2, 1 - h_2]$ , we have  $B_\alpha = \int_{-1}^1 u^2 k_2(u) du$  by the symmetry of the kernel. Further, it can be shown that the denominator (which is positive by the Cauchy-Schwarz inequality) in the definition of  $B_\alpha$  is a strictly increasing function of  $\alpha \in [0, 1]$  and hence its infimum is achieved at  $\alpha = 0$ , where it takes the value  $\int_0^1 u^2 k_2(u) du \int_0^1 k_2(u) du - (\int_0^1 u k_2(u) du)^2 =: a_0 > 0$  (again by the Cauchy-Schwarz inequality) for any non-degenerate  $k_2$ . Thus  $\sup_{\alpha \in [0,1]} |B_\alpha| \leq \sup_{\alpha \in [0,1]} |\mu_{2,\alpha}^2 - \mu_{1,\alpha} \mu_{3,\alpha}| / a_0 < \infty$  as the numerator is uniformly bounded in  $\alpha$ . Hence,  $\|\tilde{Y}_{i,w} - \tilde{X}_i\|_\infty \leq 2^{-1} \|\tilde{X}_i''\|_\infty \{h_2^2 \sup_{\alpha \in [0,1]} |B_\alpha| + O((rh_2)^{-1})\} + \|\tilde{X}_i^{(3)}\|_\infty o(h_2^2) \leq \|\tilde{X}_i''\|_\infty \{O(h_2^2) + O((rh_2)^{-1})\} + \|\tilde{X}_i^{(3)}\|_\infty o(h_2^2)$ , where the  $O(1)$  and the  $o(1)$  terms are non-random (and hence do not depend on  $i$ ).

We next control  $E\{\|\tilde{Z}_{i,w}\|_\infty^2\}$ . Observe that this does not depend on  $\tilde{X}_i$  and hence does not depend

on  $i$  (the errors are i.i.d.). Now,

$$\begin{aligned}
& E \left\{ \sup_{t \in [0,1]} \left| \frac{1}{r} \sum_{j=1}^r \frac{\{\hat{s}_2(t; h_2) - \hat{s}_1(t; h_2)(t_j - t)\} k_{2,h_2}(t_j - t) \epsilon_{ij}}{\hat{s}_2(t; h_2) \hat{s}_0(t; h_2) - \hat{s}_1^2(t; h_2)} \right|^2 \right\} \\
& \leq E \left\{ \sup_{t \in [0,1]} \frac{1}{r^2} \sum_{j=1}^r \frac{\{\hat{s}_2(t; h_2) - \hat{s}_1(t; h_2)(t_j - t)\}^2 k_{2,h_2}^2(t_j - t) \epsilon_{ij}^2}{[\hat{s}_2(t; h_2) \hat{s}_0(t; h_2) - \hat{s}_1^2(t; h_2)]^2} \right\} + \\
& E \left\{ \frac{1}{r^2} \sum_{j \neq j'} \epsilon_{ij} \epsilon_{ij'} \sup_{t \in [0,1]} \frac{\{\hat{s}_2(t; h_2) - \hat{s}_1(t; h_2)(t_j - t)\} \{\hat{s}_2(t; h_2) - \hat{s}_1(t; h_2)(t_{j'} - t)\} k_{2,h_2}(t_j - t) k_{2,h_2}(t_{j'} - t)}{[\hat{s}_2(t; h_2) \hat{s}_0(t; h_2) - \hat{s}_1^2(t; h_2)]^2} \right\} \\
& \leq M^2 r^{-1} \sup_{t \in [0,1]} \frac{1}{r} \sum_{j=1}^r \frac{\{\hat{s}_2(t; h_2) - \hat{s}_1(t; h_2)(t_j - t)\}^2 k_{2,h_2}^2(t_j - t)}{[\hat{s}_2(t; h_2) \hat{s}_0(t; h_2) - \hat{s}_1^2(t; h_2)]^2} \\
& = M^2 (rh_2)^{-1} \sup_{t \in [0,1]} \frac{\hat{s}_2^2(t; h_2) \tilde{s}_0(t; h_2) + \hat{s}_1^2(t; h_2) \tilde{s}_2(t; h_2) - 2\hat{s}_1(t; h_2) \hat{s}_2(t; h_2) \tilde{s}_1(t; h_2)}{[\hat{s}_2(t; h_2) \hat{s}_0(t; h_2) - \hat{s}_1^2(t; h_2)]^2}. \tag{17}
\end{aligned}$$

The second term on the right hand side of the first inequality vanishes due to the uncorrelatedness of the errors and the fact that the  $t_j$ 's are non-random. The bound for the first term follows from the a.s. boundedness of the errors, say with bound  $M$ . Here,  $\tilde{s}_l(t; h_2) = r^{-1} \sum_{j=1}^r (t_j - t)^l h_2^{-1} k_2^2\{(t_j - t)/h_2\}$ , which is a definition similar to  $\hat{s}_l(t; h_2)$  but with a new “kernel”  $k_2^2$ . As earlier, by Riemann sum approximations, we have  $\tilde{s}_l(t; h_2) = h_2^l \int_{-\alpha}^1 u^l k_2^2(u) du + O((rh_2)^{-1})$  for  $\alpha \in [0, 1]$  with the  $O(1)$  term being uniform on  $t \in [0, 1]$ . Define  $\nu_{l,\alpha} = \int_{-\alpha}^1 u^l k_2^2(u) du$ . Then,

$$\begin{aligned}
& \frac{\hat{s}_2^2(t; h_2) \tilde{s}_0(t; h_2) + \hat{s}_1^2(t; h_2) \tilde{s}_2(t; h_2) - 2\hat{s}_1(t; h_2) \hat{s}_2(t; h_2) \tilde{s}_1(t; h_2)}{[\hat{s}_2(t; h_2) \hat{s}_0(t; h_2) - \hat{s}_1^2(t; h_2)]^2} \\
& = \frac{\mu_{2,\alpha} \nu_{0,\alpha} + \mu_{1,\alpha}^2 \nu_{2,\alpha} - 2\mu_{1,\alpha} \mu_{2,\alpha} \nu_{1,\alpha}}{[\mu_{2,\alpha} \mu_{0,\alpha} - \mu_{1,\alpha}^2]^2} + O((rh_2)^{-1}) = C_\alpha + O((rh_2)^{-1}), \quad \text{say,}
\end{aligned}$$

for all  $\alpha \in [0, 1]$ , where the  $O(1)$  term is uniform over  $t \in [0, 1]$ . Note that the expression of  $C_\alpha$  is the same as the coefficient of the variance term in the general result obtained in Thm. 3.3 in [Fan and Gijbels \(1996\)](#) (with necessary adaptations). Using (17), it now follows that  $E\{\|\check{Z}_{i,w}\|_\infty^2\} \leq M\{\sup_{\alpha \in [0,1]} C_\alpha\} (rh_2)^{-1} + o((rh_2)^{-1}) = O((rh_2)^{-1})$ . Hence, using the assumptions in the theorem and the bounds on  $\|\check{Y}_{i,w} - \check{X}_i\|_\infty$  obtained earlier as well as the previous bound, it follows that

$$E\{\|\hat{X}_{i,w} - \check{X}_i\|_\infty^2\} = O(h_2^4) + O((rh_2)^{-1}), \tag{18}$$

where the  $O(1)$  terms are bounded and do not depend in  $i$ . Thus, using Markov's inequality, we have

$$n^{-1} \sum_{i=1}^n \|\hat{X}_{i,w} - \check{X}_i\|_\infty = O_P\{h_2^2 + (rh_2)^{-1/2}\}. \tag{19}$$

(c) Recall that  $\hat{X}_{i,e}^*(t) = \hat{X}_{i,w}(\hat{T}_{i,e}(t))$ . Thus, using (16) we have

$$\begin{aligned}
|\hat{X}_{i,e}^*(t) - \hat{X}_i(t)| & \leq |\hat{X}_{i,w}(\hat{T}_{i,e}(t)) - \check{X}_i(\hat{T}_{i,e}(t))| + |\check{X}_i(\hat{T}_{i,e}(t)) - \check{X}_i(\hat{T}_i(t))| \\
& \leq \|\hat{X}_{i,w} - \check{X}_i\|_\infty + \|\check{X}_i'\|_\infty \|\hat{T}_{i,e} - \hat{T}_i\|_\infty \\
\Rightarrow \|\hat{X}_{i,e}^* - \hat{X}_i\|_\infty & \leq \|\hat{X}_{i,w} - \check{X}_i\|_\infty + \text{const.} |\xi_i| \|T_i'\|_\infty \{R_r + |\xi_i|^{-\alpha} A_{i,r}^\alpha\}. \tag{20}
\end{aligned}$$

The proof of part (c) of this theorem now follows from (13), (15), (18) and part (c) of Theorem 3.

(d) Observe that by (20), we have

$$\|\bar{X}_{e*} - n^{-1} \sum_{i=1}^n \hat{X}_i\|_\infty$$

$$\leq n^{-1} \sum_{i=1}^n \|\hat{X}_{i,w} - \tilde{X}_i\|_\infty + \text{const.} \left\{ R_r \left( n^{-1} \sum_{i=1}^n |\xi_i| \|T'_i\|_\infty \right) + n^{-1} \sum_{i=1}^n |\xi_i|^{1-\alpha} \|T'_i\|_\infty A_{i,r}^\alpha \right\}.$$

The third term on the right hand side can be bounded using Hölder's inequality and (14) as earlier. The bounds on the first two terms are given by (19) and (15), respectively. The proof of this part of the theorem is now completed upon using these bounds along with part (e) of Theorem 3.

(e) For the proof of this part of theorem, we will use a decomposition of  $\widehat{\mathcal{K}}_{e*}$  similar to that of  $\widehat{\mathcal{K}}_r$  in the proof of part (f) of Theorem 3. In the same notation, we obtain the following bounds on  $W_1, W_2$  and  $W_3$ . First, note that  $\|W_1\| \leq n^{-1} \sum_{i=1}^n \|\hat{X}_{i,e}^* - \hat{X}_i\|_2^2 \leq 2n^{-1} \sum_{i=1}^n \|\hat{X}_{i,w} - \tilde{X}_i\|_\infty^2 + \text{const.} n^{-1} \sum_{i=1}^n \xi_i^2 \|T'_i\|_\infty^2 \{R_r + |\xi_i|^{-\alpha} A_{i,r}^\alpha\}^2$ . Applying Hölder's inequality and using (14), (15) and (18), we get that  $\|W_1\| = O_P\{h_2^4 + (rh_2)^{-1} + h_1^{4\alpha} + (rh_1^3)^{-\alpha}\}$ . Next, using part (d) of this theorem and part (e) of Theorem 3, it follows that  $\|W_2\| \leq \|\bar{X}_{e*} - \mu\|_2^2 \leq 2\|\bar{X}_{e*} - n^{-1} \sum_{i=1}^n \hat{X}_i\|_2^2 + 2\|n^{-1} \sum_{i=1}^n \hat{X}_i - \mu\|_2^2 = O_P\{h_1^{4\alpha} + (rh_1^3)^{-\alpha} + h_2^4 + (rh_2)^{-1} + n^{-1}\}$ . In a similar manner,  $\|W_3\| \leq 2n^{-1} \sum_{i=1}^n \|\hat{X}_{i,e}^* - \hat{X}_i\|_2 \|\hat{X}_i - \mu\|_2 = O_P\{h_2^2 + (rh_2)^{-1/2} + h_1^{2\alpha} + (rh_1^3)^{-\alpha/2}\}$  by the Cauchy-Schwarz inequality and the bounds obtained earlier. So, using part (f) of Theorem 3, we have  $\|\widehat{\mathcal{K}}_{e*} - \mathcal{K}\| = O_P\{h_2^2 + (rh_2)^{-1/2} + h_1^{2\alpha} + (rh_1^3)^{-\alpha/2} + n^{-1/2}\}$ . The bounds for the leading eigenvalue and eigenfunction follow directly by standard bounds in the theory of perturbation of operators.  $\square$

*Proof of Theorem 6.* First assume that  $\mu' \neq 0$ . Then, define  $G(t) = \int_0^t |\gamma_1^{-1} \mu'(u)| du / \int_0^1 |\gamma_1^{-1} \mu'(u)| du = \int_0^t |\mu'(u)| du / \int_0^1 |\mu'(u)| du$  and  $\tilde{G}_i(t) = G(T_i^{-1}(t))$  for  $t \in [0, 1]$  and  $i = 1, 2, \dots, n$ . Some algebraic manipulations yield

$$\begin{aligned} & |F_i(t) - G(t)| \\ & \leq \frac{\int_0^t |Y_{i1} \phi'_1(u) + \eta Y_{i2} \phi'_2(u)| du}{\int_0^1 |\gamma_1^{-1} \mu'(u) + Y_{i1} \phi'_1(u) + \eta Y_{i2} \phi'_2(u)| du} + \left| \frac{\int_0^t |\gamma_1^{-1} \mu'(u)| du}{\int_0^1 |\gamma_1^{-1} \mu'(u) + Y_{i1} \phi'_1(u) + \eta Y_{i2} \phi'_2(u)| du} - \frac{\int_0^t |\gamma_1^{-1} \mu'(u)| du}{\int_0^1 |\gamma_1^{-1} \mu'(u)| du} \right| \\ & \leq \frac{2 \int_0^1 |Y_{i1} \phi'_1(u) + \eta Y_{i2} \phi'_2(u)| du}{\int_0^1 |\gamma_1^{-1} \mu'(u) + Y_{i1} \phi'_1(u) + \eta Y_{i2} \phi'_2(u)| du} = Z_i. \end{aligned}$$

Thus,  $\|F_i - G\|_\infty \leq Z_i$  almost surely for each  $i$ . So  $\|\tilde{F}_i - \tilde{G}_i\|_\infty = \sup_{t \in [0,1]} |F_i(T_i^{-1}(t)) - G(T_i^{-1}(t))| = \sup_{t \in [0,1]} |F_i(t) - G(t)| \leq Z_i$ , where the last equality holds because  $T_i$  is a bijection on  $[0, 1]$ .

Next, let  $c_i = F_i^{-1}(t)$  and  $c = G^{-1}(t)$ . So,  $t = F_i(c_i) = G(c)$ . Also,  $G(c) - G(c_i) = G(c) - F_i(c_i) + F_i(c_i) - G(c_i) = F_i(c_i) - G(c_i)$  so that  $|G(c) - G(c_i)| \leq \|F_i - G\|_\infty \leq Z_i$ . The conditions of the theorem and arguments as in Lemma 2 earlier show that  $G^{-1}$  is  $\alpha$ -Hölder continuous for  $\alpha = \epsilon/(1 + \epsilon)$ . Thus, for a finite, positive constant  $C_\mu$ , we have

$$|F_i^{-1}(t) - G^{-1}(t)| = |c_i - c| = |G^{-1}(G(c_i)) - G^{-1}(G(c))| \leq C_\mu |G(c_i) - G(c)|^\alpha \leq C_\mu Z_i^\alpha.$$

Thus,  $\|F_i^{-1} - G^{-1}\|_\infty \leq C_\mu Z_i^\alpha$  almost surely. Consequently,  $\|\tilde{F}_i^{-1} - \tilde{G}_i^{-1}\|_\infty = \sup_{t \in [0,1]} |T_i(F_i^{-1}(t)) - T_i(G^{-1}(t))| \leq \|T'_i\|_\infty \|F_i^{-1} - G^{-1}\|_\infty \leq C_\mu \|T'_i\|_\infty Z_i^\alpha$  almost surely. Further,

$$\|\hat{F}^{-1} - \hat{G}^{-1}\|_\infty \leq \frac{1}{n} \sum_{i=1}^n \|\tilde{F}_i^{-1} - \tilde{G}_i^{-1}\|_\infty \leq \frac{C_\mu}{n} \sum_{i=1}^n \|T'_i\|_\infty Z_i^\alpha \leq 2C_\mu E(\|T'_1\|_\infty) E(Z_1^\alpha),$$

as  $n \rightarrow \infty$  almost surely. Here, the last inequality follows from the moment assumptions in the theorem, the Cauchy-Schwarz inequality, the strong law of large numbers and the fact that the  $Y_{il}$ 's (and hence the  $X_i$ 's) are independent of the  $T_i$ 's. Thus,

$$|\hat{T}_i^{-1}(t) - T_i(t)| = |\hat{F}^{-1}(F_i(T_i^{-1}(t))) - T_i^{-1}(t)|$$



$$\begin{aligned}
&\leq |\hat{F}^{-1}(F_i(T_i^{-1}(t))) - \hat{G}^{-1}(F_i(T_i^{-1}(t)))| + |\hat{G}^{-1}(F_i(T_i^{-1}(t))) - \hat{G}^{-1}(G(T_i^{-1}(t)))| \\
&\quad + |\hat{G}^{-1}(G(T_i^{-1}(t))) - T_i^{-1}(t)| \\
&\leq \|\hat{F}^{-1} - \hat{G}^{-1}\|_\infty + |\bar{T}(G^{-1}(F_i(T_i^{-1}(t)))) - \bar{T}(G^{-1}(G(T_i^{-1}(t))))| \\
&\quad + |\bar{T}(G^{-1}(G(T_i^{-1}(t)))) - T_i^{-1}(t)| \\
&\leq \|\hat{F}^{-1} - \hat{G}^{-1}\|_\infty + \|\bar{T}'\|_\infty C_\mu |F_i(T_i^{-1}(t)) - G(T_i^{-1}(t))|^\alpha \\
&\quad + |\bar{T}(T_i^{-1}(t)) - T_i^{-1}(t)| \\
&\leq \|\hat{F}^{-1} - \hat{G}^{-1}\|_\infty + C_\mu n^{-1} \left\{ \sum_{j=1}^n \|T_j'\|_\infty \right\} \|F_i - G\|_\infty + \|\bar{T} - Id\|_\infty \\
&\leq \text{const.} \{E(Z_1^\alpha) + Z_i + \|\bar{T} - Id\|_\infty\}, \\
\Rightarrow \|\hat{T}_i^{-1} - T_i^{-1}\|_\infty &\leq \text{const.} \{E(Z_1^\alpha) + Z_i + \|\bar{T} - Id\|_\infty\}
\end{aligned}$$

as  $n \rightarrow \infty$  almost surely, where the constant term is uniform in  $i$ .

Next, let  $t = \hat{F}^{-1}(u)$ . Then,  $n^{-1} \sum_{i=1}^n T_i(F_i^{-1}(u)) = t$ . Let  $t_* = n^{-1} \sum_{i=1}^n T_i(G^{-1}(u)) = \bar{T}(G^{-1}(u)) = \hat{G}^{-1}(u)$  so that  $u = \hat{G}(t_*)$ . Note that  $\hat{F}(t) - \hat{G}(t) = \hat{F}(t) - \hat{G}(t_*) + \hat{G}(t_*) - \hat{G}(t) = \hat{G}(t_*) - \hat{G}(t) = G(\bar{T}^{-1}(t_*)) - G(\bar{T}^{-1}(t))$ . Thus, using the assumptions in the theorem and arguments similar to those used in the proof of part (b) of Theorem 2, we have

$$\begin{aligned}
|\hat{F}(t) - \hat{G}(t)| &= |G(\bar{T}^{-1}(t_*)) - G(\bar{T}^{-1}(t))| \leq \|G'\|_\infty |\bar{T}^{-1}(t_*) - \bar{T}^{-1}(t)| \\
&\leq \|G'\|_\infty \delta^{-1} |t_* - t| \\
&\leq \|G'\|_\infty \delta^{-1} n^{-1} \sum_{i=1}^n |T_i(F_i^{-1}(u)) - T_i(G^{-1}(u))| \\
&\leq \|G'\|_\infty \delta^{-1} C_\mu n^{-1} \sum_{i=1}^n \|T_i'\|_\infty Z_i^\alpha \leq \text{const.} E(\|T_1'\|_\infty) E(Z_1^\alpha) \\
\Rightarrow \|\hat{F} - \hat{G}\|_\infty &\leq \text{const.} E(Z_1^\alpha)
\end{aligned}$$

as  $n \rightarrow \infty$  almost surely. Therefore,

$$\begin{aligned}
|\hat{T}_i(t) - T_i(t)| &= |\tilde{F}_i^{-1}(\hat{F}(t)) - T_i(t)| \\
&\leq |\tilde{F}_i^{-1}(\hat{F}(t)) - \tilde{G}_i^{-1}(\hat{F}(t))| + |\tilde{G}_i^{-1}(\hat{F}(t)) - \tilde{G}_i^{-1}(\hat{G}(t))| + |\tilde{G}_i^{-1}(\hat{G}(t)) - T_i(t)| \\
&\leq \|\tilde{F}_i^{-1} - \tilde{G}_i^{-1}\|_\infty + |T_i(G^{-1}(\hat{F}(t))) - T_i(G^{-1}(\hat{G}(t)))| + |T_i(G^{-1}(\hat{G}(t))) - T_i(t)| \\
&\leq \|\tilde{F}_i^{-1} - \tilde{G}_i^{-1}\|_\infty + \|T_i'\|_\infty C_\mu |\hat{F}(t) - \hat{G}(t)|^\alpha + |T_i(\bar{T}^{-1}(t)) - T_i(t)| \\
&\leq \|\tilde{F}_i^{-1} - \tilde{G}_i^{-1}\|_\infty + \|T_i'\|_\infty C_\mu \|\hat{F} - \hat{G}\|^\alpha + \|T_i'\|_\infty \|\bar{T}^{-1} - Id\|_\infty \\
&= \|\tilde{F}_i^{-1} - \tilde{G}_i^{-1}\|_\infty + \|T_i'\|_\infty C_\mu \|\hat{F} - \hat{G}\|^\alpha + \|T_i'\|_\infty \|\bar{T} - Id\|_\infty \\
&\leq \text{const.} \|T_i'\|_\infty \{Z_i^\alpha + E^\alpha(Z_1^\alpha) + \|\bar{T} - Id\|_\infty\} \\
\Rightarrow \|\hat{T}_i - T_i\|_\infty &\leq \text{const.} \|T_i'\|_\infty \{Z_i^\alpha + E^\alpha(Z_1^\alpha) + \|\bar{T} - Id\|_\infty\}
\end{aligned}$$

as  $n \rightarrow \infty$  almost surely, where the constant term is uniform in  $i$ .

Next, note that  $\hat{X}_i = \tilde{X}_i \circ \hat{T}_i = X_i \circ T_i^{-1} \circ \hat{T}_i = \mu \circ T_i^{-1} \circ \hat{T}_i + \gamma_1 Y_{i1} \phi_1 \circ T_i^{-1} \circ \hat{T}_i + \gamma_2 Y_{i2} \phi_2 \circ T_i^{-1} \circ \hat{T}_i$ . So,

$$\begin{aligned}
|\hat{X}_i(t) - X_i(t)| &\leq |\mu(T_i^{-1}(\hat{T}_i(t))) - \mu(t)| + \gamma_1 |Y_{i1}| |\phi_1(T_i^{-1}(\hat{T}_i(t))) - \phi_1(t)| \\
&\quad + \gamma_2 |Y_{i2}| |\phi_2(T_i^{-1}(\hat{T}_i(t))) - \phi_2(t)| \\
&\leq |T_i^{-1}(\hat{T}_i(t)) - t| \{ \|\mu'\|_\infty + \gamma_1 |Y_{i1}| \|\phi_1'\|_\infty + \gamma_1 |Y_{i2}| \|\phi_2'\|_\infty \}
\end{aligned}$$



$$\begin{aligned} \Rightarrow \|\hat{X}_i - X_i\|_\infty &\leq \|\hat{T}_i^{-1} - T_i^{-1}\|_\infty \{ \|\mu'\|_\infty + \gamma_1 |Y_{i1}| \|\phi'_1\|_\infty + \gamma_1 |Y_{i2}| \|\phi'_2\|_\infty \} \\ &\leq O_P(1) \{ E(Z_1^\alpha) + Z_i + \|\bar{T} - Id\|_\infty \} \end{aligned}$$

as  $n \rightarrow \infty$  almost surely, where the  $O_P(1)$  term is independent on  $n$ .

Next, consider the case when  $\mu' = 0$ . Then, define  $G(t) = \int_0^t |\phi'_1(u)| du / \int_0^1 |\phi'_1(u)| du$ . Some algebraic manipulations yield

$$\begin{aligned} |F_i(t) - G(t)| &= \left| \frac{\int_0^t |Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)| du}{\int_0^1 |Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)| du} - \frac{\int_0^t |\phi'_1(u)| du}{\int_0^1 |\phi'_1(u)| du} \right| \\ &\leq \frac{2\eta \int_0^1 |Y_{i2}\phi'_2(u)| du}{\int_0^1 |Y_{i1}\phi'_1(u) + \eta Y_{i2}\phi'_2(u)| du} = Z_i. \end{aligned}$$

Similar arguments as in the case of  $\mu' \neq 0$  now yield the error bounds on the estimators □

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